Creating Reswitching Examples Maple Worksheet

1.0 Introduction

This worksheet presents some symbolic computations. These computations demonstrate the validity of the equations in my paper "Creating Two-Good Reswitching Examples". This worksheet was created in Maple 7.00 on a Macintosh under Mac OS 9.0.

2.0 A Two-Commodity Model

```
> with(LinearAlgebra)
```

[*Add*, *Adjoint*, *BackwardSubstitute*, *BandMatrix*, *Basis*, *BezoutMatrix*,
BidiagonalForm, *BilinearForm*, *CharacteristicMatrix*, *CharacteristicPolynomial*,
Column, *ColumnDimension*, *ColumnOperation*, *ColumnSpace*,
CompanionMatrix, *ConditionNumber*, *ConstantMatrix*, *ConstantVector*,
CreatePermutation, *CrossProduct*, *DeleteColumn*, *DeleteRow*, *Determinant*,
DiagonalMatrix, *Dimension*, *Dimensions*, *DotProduct*, *Eigenvalues*,
Eigenvectors, *Equal*, *ForwardSubstitute*, *FrobeniusForm*, *GaussianElimination*,
GenerateEquations, *GenerateMatrix*, *GetResultDataType*, *GetResultShape*,
GivensRotationMatrix, *GramSchmidt*, *HankelMatrix*, *HermiteForm*,
HermitianTranspose, *HessenbergForm*, *HilbertMatrix*, *HouseholderMatrix*,
IdentityMatrix, *IntersectionBasis*, *IsDefinite*, *IsOrthogonal*, *IsSimilar*, *IsUnitary*,
JordanBlockMatrix, *JordanForm*, *LA_Main*, *LUDecomposition*, *LeastSquares*,
LinearSolve, *Map*, *Map2*, *MatrixAdd*, *MatrixInverse*, *MatrixMatrixMultiply*,
MatrixNorm, *MatrixScalarMultiply*, *MatrixVectorMultiply*, *MinimalPolynomial*,
Minor, *Multiply*, *NoUserValue*, *Norm*, *Normalize*, *NullSpace*,
OuterProductMatrix, *Permanent*, *Pivot*, *QRDecomposition*, *RandomMatrix*,
RandomVector, *Rank*, *ReducedRowEchelonForm*, *Row*, *RowDimension*,
RowOperation, *RowSpace*, *ScalarMatrix*, *ScalarMultiply*, *ScalarVector*,
SchurForm, *SingularValues*, *SmithForm*, *SubMatrix*, *SubVector*, *SumBasis*,
SylvesterMatrix, *ToeplitzMatrix*, *Trace*, *Transpose*, *TridiagonalForm*,
UnitVector, *VandermondeMatrix*, *VectorAdd*, *VectorAngle*, *VectorMatrixMultiply*,
VectorNorm, *VectorScalarMultiply*, *ZeroMatrix*, *ZeroVector*, *Zip*]

Define a couple of matrices:

> $A := \langle \langle a_{1,1} \mid a_{1,2} \rangle, \langle a_{2,1} \mid a_{2,2} \rangle \rangle$

$$A := \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix}$$

> $a0 := \langle a01 \mid a02 \rangle$

$$a0 := [a01, a02]$$

Consider Equations 8 and 9 in the paper:

> $w := \frac{(a_{1,1} a_{2,2} - a_{1,2} a_{2,1}) R^2 - (a_{1,1} + a_{2,2}) R + 1}{(a_{2,1} a02 - a_{2,2} a01) R + a01}$

$$w := \frac{(a_{1,1} a_{2,2} - a_{1,2} a_{2,1}) R^2 - (a_{1,1} + a_{2,2}) R + 1}{(a_{2,1} a02 - a_{2,2} a01) R + a01}$$

> $p_2 := \frac{(a_{1,2} a01 - a_{1,1} a02) R + a02}{(a_{2,1} a02 - a_{2,2} a01) R + a01}$

$$p_2 := \frac{(a_{1,2} a01 - a_{1,1} a02) R + a02}{(a_{2,1} a02 - a_{2,2} a01) R + a01}$$

I want to show these equations satisfy the system of two equations in Display 8, where

> $p_1 := 1$

$$p_1 := 1$$

Calculate the difference between the left-hand and right-hand sides of Display 8:

```
> simplify(VectorAdd(VectorAdd(
    VectorScalarMultiply(VectorMatrixMultiply(<p1 | p2>, A), R),
    VectorScalarMultiply(a0, w)), VectorScalarMultiply(<p1 | p2>, -1)))
```

$$[0, 0]$$

So Equation 8 holds. Here's another demonstration:

```
> simplify(VectorAdd(VectorMatrixMultiply(<R p1 | R p2>, A),
    VectorScalarMultiply(a0, w)))
```

$$\left[1, -\frac{R a_{1,2} a01 - R a_{1,1} a02 + a02}{-R a_{2,1} a02 + R a_{2,2} a01 - a01} \right]$$

Now I want to show that when R is given by Equation 11, the wage specified by Equation 9 is zero.

> *restart*

Here's equation 11:

> $R_{max} := \frac{2}{a_{1,1} + a_{2,2} + \sqrt{(a_{1,1} - a_{2,2})^2 + 4 a_{1,2} a_{2,1}}}$

$$R_{max} := 2 \frac{1}{a_{1,1} + a_{2,2} + \sqrt{a_{1,1}^2 - 2 a_{1,1} a_{2,2} + a_{2,2}^2 + 4 a_{1,2} a_{2,1}}}$$

Here's another specification of Equation 9:

> $w := R \rightarrow \frac{(a_{1,1} a_{2,2} - a_{1,2} a_{2,1}) R^2 - (a_{1,1} + a_{2,2}) R + 1}{(a_{2,1} a_{0,2} - a_{2,2} a_{0,1}) R + a_{0,1}}$

$$w := R \rightarrow \frac{(a_{1,1} a_{2,2} - a_{1,2} a_{2,1}) R^2 - (a_{1,1} + a_{2,2}) R + 1}{(a_{2,1} a_{0,2} - a_{2,2} a_{0,1}) R + a_{0,1}}$$

If the mathematics is correct, the next should be zero:

> simplify(w(R_{max}))

$$0$$

3.0 Given Points on the Wage-Profits Curve

Set the coefficients of production as specified in Equations 17 through 21:

> *restart*

> $a_{0,\,1} := 1$

$$a_{0,\,1} := 1$$

> $a_{0,\,2} := 1$

$$a_{0,\,2} := 1$$

> $c_1 := (R_2 - R_1)(R_1 - 1)(R_2 - 1)$

$$c_1 := (R_2 - R_1)(R_1 - 1)(R_2 - 1)$$

> $c_2 := (w_{max} - w_1)(R_2 - 1) - (w_{max} - w_2)(R_1 - 1)$

$$c_2 := (w_{max} - w_1)(R_2 - 1) - (w_{max} - w_2)(R_1 - 1)$$

> $a_{1,\,1} := \frac{(R_1 w_{max} - w_1)(w_2 - 1)}{c_2 R_2} - \frac{(R_2 w_{max} - w_2)(w_1 - 1)}{c_2 R_1} + \frac{(R_2 w_1 - R_1 w_2)(w_{max} - 1)}{c_2} - a_{2,\,2}$

$$a_{1,\,1} := \frac{(R_1 w_{max} - w_1)(w_2 - 1)}{((w_{max} - w_1)(R_2 - 1) - (w_{max} - w_2)(R_1 - 1))R_2} - \frac{(R_2 w_{max} - w_2)(w_1 - 1)}{((w_{max} - w_1)(R_2 - 1) - (w_{max} - w_2)(R_1 - 1))R_1} + \frac{(R_2 w_1 - R_1 w_2)(w_{max} - 1)}{(w_{max} - w_1)(R_2 - 1) - (w_{max} - w_2)(R_1 - 1)} - a_{2,\,2}$$

> $a_{1,\,2} := (R_2 (w_1 - 1)(w_{max} - w_2) - R_1 (w_2 - 1)(w_{max} - w_1) - R_1 R_2 (w_{max} - 1)(w_1 - w_2) + a_{2,\,2} R_2 (R_1 R_2 (w_{max} - 1) w_1 - (R_2 w_{max} - w_2)(w_1 - 1)) - a_{2,\,2} R_1 (R_1 R_2 (w_{max} - 1) w_2 - (R_1 w_{max} - w_1)(w_2 - 1)) - a_{2,\,2}^2 c_2 R_1 R_2) \Big/ (R_2 ((R_2 - 1) w_1 - R_1 R_2 w_{max}) - R_1 ((R_1 - 1) w_2 - R_1 R_2 w_{max}) + c_1 + c_2 R_1 R_2 a_{2,\,2})$

$$a_{1,\,2} := (R_2 (w_1 - 1)(w_{max} - w_2) - R_1 (w_2 - 1)(w_{max} - w_1) - R_1 R_2 (w_{max} - 1)(w_1 - w_2) + a_{2,\,2} R_2 (R_1 R_2 (w_{max} - 1) w_1 - (R_2 w_{max} - w_2)(w_1 - 1)) - a_{2,\,2} R_1 (R_1 R_2 (w_{max} - 1) w_2 - (R_1 w_{max} - w_1)(w_2 - 1)) - a_{2,\,2}^2 ((w_{max} - w_1)(R_2 - 1) - (w_{max} - w_2)(R_1 - 1)) R_1 R_2) \Big/ (R_2 ((R_2 - 1) w_1 - R_1 R_2 w_{max}) - R_1 ((R_1 - 1) w_2 - R_1 R_2 w_{max}) + (R_2 - R_1)(R_1 - 1)(R_2 - 1)$$

$$+ ((w_{max} - w_1)(R_2 - 1) - (w_{max} - w_2)(R_1 - 1)) R_1 R_2 a_{2,2})$$

$$> a_{2,1} := a_{2,2} - \frac{(R_1 - 1) w_2}{c_2 R_2} + \frac{(R_2 - 1) w_1}{c_2 R_1} - \frac{(R_2 - R_1) w_{max}}{c_2} + \frac{c_1}{c_2 R_1 R_2}$$

$$a_{2,1} := a_{2,2} - \frac{(R_1 - 1) w_2}{((w_{max} - w_1)(R_2 - 1) - (w_{max} - w_2)(R_1 - 1)) R_2}$$
$$+ \frac{(R_2 - 1) w_1}{((w_{max} - w_1)(R_2 - 1) - (w_{max} - w_2)(R_1 - 1)) R_1}$$
$$- \frac{(R_2 - R_1) w_{max}}{(w_{max} - w_1)(R_2 - 1) - (w_{max} - w_2)(R_1 - 1)}$$
$$+ \frac{(R_2 - R_1)(R_1 - 1)(R_2 - 1)}{((w_{max} - w_1)(R_2 - 1) - (w_{max} - w_2)(R_1 - 1)) R_1 R_2}$$

I now show that these coefficients give a wage-profits curve that goes through the two given switch points and the given maximum wage.

$$> w := R \rightarrow \frac{(a_{1,1} a_{2,2} - a_{1,2} a_{2,1}) R^2 - (a_{1,1} + a_{2,2}) R + 1}{(a_{2,1} a_{0,2} - a_{2,2} a_{0,1}) R + a_{0,1}}$$

$$w := R \rightarrow \frac{(a_{1,1} a_{2,2} - a_{1,2} a_{2,1}) R^2 - (a_{1,1} + a_{2,2}) R + 1}{(a_{2,1} a_{0,2} - a_{2,2} a_{0,1}) R + a_{0,1}}$$

> simplify(w(R_1))

$$w_1$$

> simplify(w(R_2))

$$w_2$$

> simplify(w(1))

$$w_{max}$$

So the equations for the coefficients of production define a wage-profits curve with the desired switch points and maximum wage.

Let $x = a_{1,2} a_{2,1}$. According to Equation 22:

$$> x := \frac{R_2 (w_{max} - w_2)(w_1 - 1) - R_1 (w_{max} - w_1)(w_2 - 1)}{c_2 R_1 R_2}$$
$$- \frac{(w_{max} - 1)(w_1 - w_2)}{c_2}$$
$$+ \frac{(R_1 (R_1 w_{max} - w_1)(w_2 - 1) - R_2 (R_2 w_{max} - w_2)(w_1 - 1)) a_{2,2}}{c_2 R_1 R_2}$$
$$- \frac{(R_1 w_2 - R_2 w_1)(w_{max} - 1) a_{2,2}}{c_2} - a_{2,2}^2$$

$$x := \frac{R_2 (w_1 - 1)(w_{max} - w_2) - R_1 (w_2 - 1)(w_{max} - w_1)}{((w_{max} - w_1)(R_2 - 1) - (w_{max} - w_2)(R_1 - 1)) R_1 R_2}$$
$$- \frac{(w_{max} - 1)(w_1 - w_2)}{(w_{max} - w_1)(R_2 - 1) - (w_{max} - w_2)(R_1 - 1)}$$
$$+ \frac{((R_1 w_{max} - w_1)(w_2 - 1) R_1 - (R_2 w_{max} - w_2)(w_1 - 1) R_2) a_{2,2}}{((w_{max} - w_1)(R_2 - 1) - (w_{max} - w_2)(R_1 - 1)) R_1 R_2}$$

$$-\frac{(R_1 w_2 - R_2 w_1)(w_{max} - 1) a_{2,2}}{(w_{max} - w_1)(R_2 - 1) - (w_{max} - w_2)(R_1 - 1)} - a_{2,2}^2$$

Check Equation 22:

> simplify($x - a_{1,2} a_{2,1}$)

$$0$$

Let $y = (1 - a_{1,1})(1 - a_{2,2}) - a_{1,2} a_{2,1}$. According to Equation 23:

> $y := \left(\frac{w_2}{c_2 R_2} - \frac{w_1}{c_2 R_1}\right)(R_1 - 1)(R_2 - 1) w_{max} + \frac{c_1 w_{max}}{c_2 R_1 R_2}$

$$y := \left(\frac{w_2}{((w_{max} - w_1)(R_2 - 1) - (w_{max} - w_2)(R_1 - 1)) R_2} - \frac{w_1}{((w_{max} - w_1)(R_2 - 1) - (w_{max} - w_2)(R_1 - 1)) R_1}\right)(R_1 - 1)(R_2 - 1)$$
$$w_{max} + \frac{(R_2 - R_1)(R_1 - 1)(R_2 - 1) w_{max}}{((w_{max} - w_1)(R_2 - 1) - (w_{max} - w_2)(R_1 - 1)) R_1 R_2}$$

Check Equation 23:

> simplify($y - (1 - a_{1,1})(1 - a_{2,2}) + a_{1,2} a_{2,1}$)

$$0$$

Now I want to show the maximum rate of profit is not a function of $a_{2,2}$. From Equation 11, the denominator of the maximum rate of profit is
$a_{1,1} + a_{2,2} + \sqrt{(a_{1,1} - a_{2,2})^2 + 4 a_{1,2} a_{2,1}}$.

The part of this denominator outside the square root does not depend on $a_{2,2}$:

> simplify($a_{1,1} + a_{2,2}$)

$$(R_1^2 w_{max} w_2 - R_1^2 w_{max} - R_1 w_1 w_2 + R_1 w_1 - R_2^2 w_{max} w_1 + R_2^2 w_{max} + R_2 w_1 w_2 - R_2 w_2 + R_2^2 R_1 w_{max} w_1 - R_2^2 R_1 w_1 - R_2 R_1^2 w_{max} w_2 + R_2 R_1^2 w_2) \Big/ ($$
$$(R_2 w_{max} - R_2 w_1 + w_1 - R_1 w_{max} + R_1 w_2 - w_2) R_2 R_1)$$

Consider the quantity whose square root is sought:

> simplify($(a_{1,1} - a_{2,2})^2 + 4 a_{1,2} a_{2,1}$)

$$(-2 R_2^2 R_1 w_{max} w_1 - 2 R_2 R_1^2 w_{max} w_2 + R_1^4 w_{max}^2 + R_1^2 w_1^2$$
$$+ R_2^4 w_{max}^2 + R_2^2 w_2^2 + R_1^4 w_{max}^2 w_2^2 - 2 R_1^4 w_{max}^2 w_2$$
$$- 2 R_1^3 w_{max} w_1 + 6 R_1^2 w_{max}^2 R_2^2 + R_1^2 w_1^2 w_2^2 - 2 R_1^2 w_1^2 w_2$$
$$+ 6 R_1^2 w_1^2 R_2^2 + R_2^4 w_{max}^2 w_1^2 - 2 R_2^4 w_{max}^2 w_1 - 2 R_2^3 w_{max} w_2$$
$$+ R_2^2 w_1^2 w_2^2 - 2 R_2^2 w_1 w_2^2 + 6 R_2^2 w_2^2 R_1^2 + R_2^4 R_1^2 w_1^2$$
$$+ R_2^2 R_1^4 w_2^2 - 2 R_1^3 w_{max} w_2^2 w_1 + 4 R_1^3 w_{max} w_2 w_1$$
$$- 2 R_1^2 w_{max}^2 w_2 R_2^2 + 2 R_1^2 w_{max} w_2^2 R_2 - 2 R_1^4 w_{max}^2 w_2^2 R_2$$
$$+ 2 R_1^4 w_{max} w_2^2 R_2 - 2 R_1^2 w_{max}^2 R_2^2 w_1 + 2 R_1^3 w_{max}^2 R_2^2 w_1$$

$$-2\,R_1^3\,w_{max}\,R_2^2\,w_1 + 2\,R_1^4\,w_{max}^2\,R_2\,w_2 - 2\,R_1^4\,w_{max}\,R_2\,w_2$$
$$-2\,R_1\,w_1^2\,w_2^2\,R_2 + 2\,R_1\,w_1\,w_2^2\,R_2 - 2\,R_1^2\,w_1^2\,w_2\,R_2^2$$
$$+2\,R_1^3\,w_1\,w_2^2\,R_2 + 2\,R_1\,w_1^2\,R_2^2\,w_{max} + 2\,R_1\,w_1^2\,R_2\,w_2$$
$$-2\,R_1\,w_1\,R_2\,w_2 - 2\,R_1^2\,w_1^2\,R_2^2\,w_{max} - 2\,R_1^3\,w_1\,R_2\,w_2$$
$$-2\,R_2^3\,w_{max}\,w_1^2\,w_2 + 4\,R_2^3\,w_{max}\,w_1\,w_2 - 2\,R_2^4\,w_{max}^2\,w_1^2\,R_1$$
$$+2\,R_2^4\,w_{max}\,w_1^2\,R_1 + 2\,R_2^4\,w_{max}^2\,R_1\,w_1 - 2\,R_2^4\,w_{max}\,R_1\,w_1$$
$$+2\,R_2^3\,w_{max}^2\,R_1^2\,w_2 - 2\,R_2^3\,w_{max}\,R_1^2\,w_2 + 2\,R_2^3\,w_1^2\,w_2\,R_1$$
$$-2\,R_2^2\,w_1\,w_2^2\,R_1^2 - 2\,R_2^3\,w_2\,R_1\,w_1 - 2\,R_2^2\,w_2^2\,R_1^2\,w_{max}$$
$$+R_2^4\,R_1^2\,w_{max}^2\,w_1^2 - 2\,R_2^4\,R_1^2\,w_{max}\,w_1^2 - 2\,R_2^3\,R_1^3\,w_1\,w_2$$
$$+R_2^2\,R_1^4\,w_{max}^2\,w_2^2 - 2\,R_2^2\,R_1^4\,w_{max}\,w_2^2 + 24\,R_2^2\,R_1^2\,w_{max}\,w_1\,w_2$$
$$-4\,R_2^3\,R_1\,w_{max}^2 + 4\,R_2^3\,R_1^2\,w_1^2\,w_{max} - 8\,R_2^2\,R_1^2\,w_1\,w_2$$
$$+4\,R_2\,R_1^3\,w_{max}\,w_2 + 4\,R_2^2\,R_1^3\,w_1\,w_2 + 4\,R_2^3\,R_1^2\,w_2\,w_1$$
$$+4\,R_2^3\,R_1\,w_{max}\,w_1 + 4\,R_2\,R_1^3\,w_1\,w_{max} + 4\,R_2^3\,R_1\,w_2\,w_{max}$$
$$+4\,R_2\,R_1^3\,w_{max}^2\,w_2 - 4\,R_2\,R_1^3\,w_{max}\,w_2^2 + 4\,R_2^3\,R_1\,w_{max}^2\,w_1$$
$$-4\,R_2^3\,R_1\,w_{max}\,w_1^2 + 4\,R_2\,R_1^2\,w_1^2\,w_2 - 4\,R_2\,R_1^2\,w_1\,w_2^2$$
$$-4\,R_2^2\,R_1\,w_1^2\,w_2 + 4\,R_2^2\,R_1\,w_1\,w_2^2 - 4\,R_2^3\,R_1^2\,w_{max}^2\,w_1$$
$$-4\,R_2^2\,R_1^3\,w_{max}^2\,w_2 - 4\,R_2\,R_1^2\,w_1^2 - 4\,R_2\,R_1^3\,w_{max}^2 - 4\,R_2^2\,R_1\,w_2^2$$
$$-4\,R_2^3\,R_1^2\,w_1^2 - 4\,R_2^2\,R_1^3\,w_2^2 + 4\,R_2\,R_1^2\,w_1\,w_{max} + 4\,R_2\,R_1^2\,w_1\,w_2$$
$$+4\,R_2^2\,R_1\,w_2\,w_{max} + 4\,R_2^2\,R_1\,w_2\,w_1 + 4\,R_2^3\,R_1^2\,w_{max}\,w_1$$
$$+4\,R_2^2\,R_1^3\,w_{max}\,w_2 - 8\,R_2^2\,R_1^2\,w_{max}\,w_1 - 8\,R_2^2\,R_1^2\,w_{max}\,w_2$$
$$+4\,R_2^2\,R_1^3\,w_{max}\,w_2^2 - 2\,R_1^2\,w_{max}^2\,w_2\,R_2^2\,w_1$$
$$+2\,R_1^2\,w_{max}\,w_2^2\,R_2\,w_1 + 2\,R_1^3\,w_{max}^2\,w_2\,R_2^2\,w_1$$
$$-6\,R_1^3\,w_{max}\,w_2\,R_2^2\,w_1 - 6\,R_1^2\,w_{max}\,R_2\,w_1\,w_2$$
$$+2\,R_1\,w_1^2\,w_2\,R_2^2\,w_{max} - 6\,R_1\,w_1\,w_2\,R_2^2\,w_{max}$$
$$-2\,R_1^2\,w_1^2\,w_2\,R_2^2\,w_{max} + 2\,R_1^3\,w_1\,w_2^2\,R_2\,w_{max}$$
$$-6\,R_1^3\,w_1\,R_2\,w_{max}\,w_2 + 2\,R_2^3\,w_{max}^2\,w_1\,R_1^2\,w_2$$
$$-6\,R_2^3\,w_{max}\,w_1\,R_1^2\,w_2 + 2\,R_2^3\,w_1^2\,w_2\,R_1\,w_{max}$$
$$-2\,R_2^2\,w_1\,w_2^2\,R_1^2\,w_{max} - 6\,R_2^3\,w_2\,R_1\,w_{max}\,w_1$$
$$-2\,R_2^3\,R_1^3\,w_{max}^2\,w_1\,w_2 + 4\,R_2^3\,R_1^3\,w_{max}\,w_1\,w_2) \Big/ ($$
$$(R_2\,w_{max} - R_2\,w_1 + w_1 - R_1\,w_{max} + R_1\,w_2 - w_2)^2\,R_2^2\,R_1^2)$$

Notice that $a_{2,2}$ does not appear in the above expression. This justifies the footnote in the text above Equation 16.

So Section 3 of the paper is validated.

> restart

4.0 Choice of Processes for Producing the Numeraire Commodity

Specify $a_{0,2}$ as unity:

> $a02 := 1$

$$a02 := 1$$

There appears to be a bug in the version of Maple I have when quotients this complicated interact with subscripted variables. The equations expressed in this and the succeeding section work around that bug.

Set the coefficients of production as specified in Equations 24 through 28

> $c1 := (R_2 - R_1)(R_1 - 1)(R_2 - 1)$

$$c1 := (R_2 - R_1)(R_1 - 1)(R_2 - 1)$$

> $c3 := (w_{max,b} - w_1)(R_2 - 1) - (w_{max,b} - w_2)(R_1 - 1)$

$$c3 := (w_{max,b} - w_1)(R_2 - 1) - (w_{max,b} - w_2)(R_1 - 1)$$

> $c4 := -a12\ a22\ c3\ R_1 R_2$
> $+ a12\,(R_2 (R_2 - 1)(w_{max,b} R_1 - w_1) - R_1 (R_1 - 1)(w_{max,b} R_2 - w_2))$
> $+ a22\,(w_1 (R_1 - 1)(w_{max,b} R_2^2 - w_2) - w_2 (R_2 - 1)(w_{max,b} R_1^2 - w_1))$
> $+ w_2 (R_2 - 1)(w_{max,b} R_1 - w_1) - w_1 (R_1 - 1)(w_{max,b} R_2 - w_2)$

$$c4 := -a12\ a22\,((w_{max,b} - w_1)(R_2 - 1) - (w_{max,b} - w_2)(R_1 - 1))\,R_1 R_2$$
$$+ a12\,(R_2 (R_2 - 1)(w_{max,b} R_1 - w_1) - R_1 (R_1 - 1)(w_{max,b} R_2 - w_2))$$
$$+ a22\,(w_1 (R_1 - 1)(w_{max,b} R_2^2 - w_2) - w_2 (R_2 - 1)(w_{max,b} R_1^2 - w_1))$$
$$+ w_2 (R_2 - 1)(w_{max,b} R_1 - w_1) - w_1 (R_1 - 1)(w_{max,b} R_2 - w_2)$$

> $b01 := (a22^2\ c3\ R_1 R_2 + a22$
> $((R_1 - 1)(R_1 + 1)(w_{max,b} - R_2 w_2) - (R_2 - 1)(R_2 + 1)(w_{max,b} - R_1 w_1))$
> $+ a12\ c1 + (R_2 - 1)(w_{max,b} - R_1 w_1) - (R_1 - 1)(w_{max,b} - R_2 w_2))\,/\,c4$

$$b01 := (a22^2\,((w_{max,b} - w_1)(R_2 - 1) - (w_{max,b} - w_2)(R_1 - 1))\,R_1 R_2 +$$
$$a22\,($$
$$(R_1 - 1)(R_1 + 1)(w_{max,b} - R_2 w_2) - (R_2 - 1)(R_2 + 1)(w_{max,b} - R_1 w_1)$$
$$) + a12\,(R_2 - R_1)(R_1 - 1)(R_2 - 1) + (R_2 - 1)(w_{max,b} - R_1 w_1)$$
$$- (R_1 - 1)(w_{max,b} - R_2 w_2))\ /\ ($$
$$-a12\ a22\,((w_{max,b} - w_1)(R_2 - 1) - (w_{max,b} - w_2)(R_1 - 1))\,R_1 R_2$$
$$+ a12\,(R_2 (R_2 - 1)(w_{max,b} R_1 - w_1) - R_1 (R_1 - 1)(w_{max,b} R_2 - w_2))$$

$$+ a22 (w_1 (R_1 - 1)(w_{max,b} R_2^2 - w_2) - w_2 (R_2 - 1)(w_{max,b} R_1^2 - w_1))$$

$$+ w_2 (R_2 - 1)(w_{max,b} R_1 - w_1) - w_1 (R_1 - 1)(w_{max,b} R_2 - w_2))$$

> b11 := (a12 a22^2 c3 R1 R2 + a12 a22 R1 (R1 (R2 + 1) (w_{max,b} - w2) - (w1 - w2)) - a12 a22 R2 (R2 (R1 + 1) (w_{max,b} - w1) + w1 - w2) + a12 ((R2 + 1) (R2 - 1) (w_{max,b} - w1) - (R1 + 1) (R1 - 1) (w_{max,b} - w2)) + a22 (w1 (R1 - 1) (R2 w_{max,b} - w2) - w2 (R2 - 1) (R1 w_{max,b} - w1)) + w2 (R2 - 1) (w_{max,b} - w1) - w1 (R1 - 1) (w_{max,b} - w2)) / c4

$$b11 := (a12\, a22^2 ((w_{max,b} - w_1)(R_2 - 1) - (w_{max,b} - w_2)(R_1 - 1)) R_1 R_2$$

$$+ a12\, a22\, R_1 (R_1 (R_2 + 1)(w_{max,b} - w_2) - w_1 + w_2)$$

$$- a12\, a22\, R_2 (R_2 (R_1 + 1)(w_{max,b} - w_1) + w_1 - w_2) +$$

$$a12 ((R_2 + 1)(R_2 - 1)(w_{max,b} - w_1) - (R_1 + 1)(R_1 - 1)(w_{max,b} - w_2))$$

$$+ a22 (w_1 (R_1 - 1)(w_{max,b} R_2 - w_2) - w_2 (R_2 - 1)(w_{max,b} R_1 - w_1))$$

$$+ w_2 (R_2 - 1)(w_{max,b} - w_1) - w_1 (R_1 - 1)(w_{max,b} - w_2)) \Big/ ($$

$$-a12\, a22 ((w_{max,b} - w_1)(R_2 - 1) - (w_{max,b} - w_2)(R_1 - 1)) R_1 R_2$$

$$+ a12 (R_2 (R_2 - 1)(w_{max,b} R_1 - w_1) - R_1 (R_1 - 1)(w_{max,b} R_2 - w_2))$$

$$+ a22 (w_1 (R_1 - 1)(w_{max,b} R_2^2 - w_2) - w_2 (R_2 - 1)(w_{max,b} R_1^2 - w_1))$$

$$+ w_2 (R_2 - 1)(w_{max,b} R_1 - w_1) - w_1 (R_1 - 1)(w_{max,b} R_2 - w_2))$$

> b21 := (1 - a22) (1 - R1 a22) (1 - R2 a22) ((R1 - 1) (w_{max,b} - w2) - (R2 - 1) (w_{max,b} - w1)) / c4

$$b21 := (1 - a22)(1 - R_1 a22)(1 - R_2 a22)$$

$$((w_{max,b} - w_2)(R_1 - 1) - (w_{max,b} - w_1)(R_2 - 1)) \Big/ ($$

$$-a12\, a22 ((w_{max,b} - w_1)(R_2 - 1) - (w_{max,b} - w_2)(R_1 - 1)) R_1 R_2$$

$$+ a12 (R_2 (R_2 - 1)(w_{max,b} R_1 - w_1) - R_1 (R_1 - 1)(w_{max,b} R_2 - w_2))$$

$$+ a22 (w_1 (R_1 - 1)(w_{max,b} R_2^2 - w_2) - w_2 (R_2 - 1)(w_{max,b} R_1^2 - w_1))$$

$$+ w_2 (R_2 - 1)(w_{max,b} R_1 - w_1) - w_1 (R_1 - 1)(w_{max,b} R_2 - w_2))$$

Now define the wage-profits curve for the specified coefficients of production:

$$> w := R \rightarrow \frac{(b11\, a22 - a12\, b21) R^2 - (b11 + a22) R + 1}{(b21\, a02 - a22\, b01) R + b01}$$

$$w := R \rightarrow \frac{(b11\, a22 - a12\, b21) R^2 - (b11 + a22) R + 1}{(b21\, a02 - a22\, b01) R + b01}$$

Show that the maxmum wage is $w_{max,b}$ and that the wage-profits curve goes through the points (R_1 , w_1) and (R_2 , w_2):

> simplify(w(1))

$$w_{max,\,b}$$

> simplify(w(R_1))

$$w_1$$

> simplify(w(R_2))

$$w_2$$

Check Equation 29. The left hand side is:

> $x := (1 - b11)\,(1 - a22) - a12\,b21$

$$
\begin{aligned}
x := (1 - (\\
& a12\,a22^2\,((w_{max,\,b} - w_1)\,(R_2 - 1) - (w_{max,\,b} - w_2)\,(R_1 - 1))\,R_1\,R_2 \\
& + a12\,a22\,R_1\,(R_1\,(R_2 + 1)\,(w_{max,\,b} - w_2) - w_1 + w_2) \\
& - a12\,a22\,R_2\,(R_2\,(R_1 + 1)\,(w_{max,\,b} - w_1) + w_1 - w_2) + \\
& a12\,((R_2 + 1)\,(R_2 - 1)\,(w_{max,\,b} - w_1) - (R_1 + 1)\,(R_1 - 1)\,(w_{max,\,b} - w_2)) \\
& + a22\,(w_1\,(R_1 - 1)\,(w_{max,\,b}\,R_2 - w_2) - w_2\,(R_2 - 1)\,(w_{max,\,b}\,R_1 - w_1)) \\
& + w_2\,(R_2 - 1)\,(w_{max,\,b} - w_1) - w_1\,(R_1 - 1)\,(w_{max,\,b} - w_2)) \Big/ (\\
& -a12\,a22\,((w_{max,\,b} - w_1)\,(R_2 - 1) - (w_{max,\,b} - w_2)\,(R_1 - 1))\,R_1\,R_2 \\
& + a12\,(R_2\,(R_2 - 1)\,(w_{max,\,b}\,R_1 - w_1) - R_1\,(R_1 - 1)\,(w_{max,\,b}\,R_2 - w_2)) \\
& + a22\,(w_1\,(R_1 - 1)\,(w_{max,\,b}\,R_2^2 - w_2) - w_2\,(R_2 - 1)\,(w_{max,\,b}\,R_1^2 - w_1)) \\
& + w_2\,(R_2 - 1)\,(w_{max,\,b}\,R_1 - w_1) - w_1\,(R_1 - 1)\,(w_{max,\,b}\,R_2 - w_2))) \\
& (1 - a22) - a12\,(1 - a22)\,(1 - R_1\,a22)\,(1 - R_2\,a22) \\
& ((w_{max,\,b} - w_2)\,(R_1 - 1) - (w_{max,\,b} - w_1)\,(R_2 - 1)) \Big/ (\\
& -a12\,a22\,((w_{max,\,b} - w_1)\,(R_2 - 1) - (w_{max,\,b} - w_2)\,(R_1 - 1))\,R_1\,R_2 \\
& + a12\,(R_2\,(R_2 - 1)\,(w_{max,\,b}\,R_1 - w_1) - R_1\,(R_1 - 1)\,(w_{max,\,b}\,R_2 - w_2)) \\
& + a22\,(w_1\,(R_1 - 1)\,(w_{max,\,b}\,R_2^2 - w_2) - w_2\,(R_2 - 1)\,(w_{max,\,b}\,R_1^2 - w_1)) \\
& + w_2\,(R_2 - 1)\,(w_{max,\,b}\,R_1 - w_1) - w_1\,(R_1 - 1)\,(w_{max,\,b}\,R_2 - w_2))
\end{aligned}
$$

The right hand side is:

> $y := w_{max,\,b}\,(R_1 - 1)\,(R_2 - 1)\,(1 - a22)$
> $(a22\,(R_2\,w_1 - R_1\,w_2) + a12\,(R_2 - R_1) - (w_1 - w_2))\,/\,c4$

$$
\begin{aligned}
y := w_{max,\,b}\,(R_1 - 1)\,(R_2 - 1)\,(1 - a22)\\
& (a22\,(w_1\,R_2 - w_2\,R_1) + a12\,(R_2 - R_1) - w_1 + w_2) \Big/ (\\
& -a12\,a22\,((w_{max,\,b} - w_1)\,(R_2 - 1) - (w_{max,\,b} - w_2)\,(R_1 - 1))\,R_1\,R_2 \\
& + a12\,(R_2\,(R_2 - 1)\,(w_{max,\,b}\,R_1 - w_1) - R_1\,(R_1 - 1)\,(w_{max,\,b}\,R_2 - w_2)) \\
& + a22\,(w_1\,(R_1 - 1)\,(w_{max,\,b}\,R_2^2 - w_2) - w_2\,(R_2 - 1)\,(w_{max,\,b}\,R_1^2 - w_1)) \\
& + w_2\,(R_2 - 1)\,(w_{max,\,b}\,R_1 - w_1) - w_1\,(R_1 - 1)\,(w_{max,\,b}\,R_2 - w_2))
\end{aligned}
$$

And these two sides are equal:

> simplify$(x-y)$

$$0$$

> restart

5.0 Choice of Processes for Producing the Non-Numeraire Commodity

Specify $a_{0,1}$ as unity:

> $a01 := 1$

$$a01 := 1$$

Some useful expressions:

> $c1 := (R_2 - R_1)(R_1 - 1)(R_2 - 1)$

$$c1 := (R_2 - R_1)(R_1 - 1)(R_2 - 1)$$

> $c3 := (w_{max,b} - w_1)(R_2 - 1) - (w_{max,b} - w_2)(R_1 - 1)$

$$c3 := (w_{max,b} - w_1)(R_2 - 1) - (w_{max,b} - w_2)(R_1 - 1)$$

Specify $b_{0,2}$ as in Equation 32:

> $b02 := 1\left(\frac{R_2 (R_1 - 1)(w_{max,b} - 1)(w_1 - 1)}{R_1\, c3} - \frac{R_1 (R_2 - 1)(w_{max,b} - 1)(w_2 - 1)}{R_2\, c3} + \frac{(R_2 - R_1)(w_1 - 1)(w_2 - 1)}{R_1 R_2\, c3} - a11\right) / a21$

$$b02 := \left(\frac{R_2 (R_1 - 1)(w_{max,b} - 1)(w_1 - 1)}{R_1 ((w_{max,b} - w_1)(R_2 - 1) - (w_{max,b} - w_2)(R_1 - 1))} - \frac{R_1 (R_2 - 1)(w_{max,b} - 1)(w_2 - 1)}{R_2 ((w_{max,b} - w_1)(R_2 - 1) - (w_{max,b} - w_2)(R_1 - 1))} + \frac{(R_2 - R_1)(w_1 - 1)(w_2 - 1)}{R_1 R_2 ((w_{max,b} - w_1)(R_2 - 1) - (w_{max,b} - w_2)(R_1 - 1))} - a11\right) / a21$$

Spcify $b_{1,2}$ as in Equation 33:

> $b12 := 1\left(\frac{R_2 (w_{max,b} - w_2)(w_1 - 1) - R_1 (w_{max,b} - w_1)(w_2 - 1)}{R_1 R_2\, c3} - \frac{(w_{max,b} - 1)(w_1 - w_2)}{c3} + \frac{(R_1 (R_1 w_{max,b} - w_1)(w_2 - 1) - R_2 (R_2 w_{max,b} - w_2)(w_1 - 1))\, a11}{R_1 R_2\, c3} - \frac{(R_1 w_2 - R_2 w_1)(w_{max,b} - 1)\, a11}{c3} - a11^2\right) / a21$

$$b12 := \left(\frac{R_2 (w_{max,b} - w_2)(w_1 - 1) - R_1 (w_{max,b} - w_1)(w_2 - 1)}{R_1 R_2 ((w_{max,b} - w_1)(R_2 - 1) - (w_{max,b} - w_2)(R_1 - 1))} - \frac{(w_{max,b} - 1)(w_1 - w_2)}{(w_{max,b} - w_1)(R_2 - 1) - (w_{max,b} - w_2)(R_1 - 1)} + \right.$$

$$(R_1 (R_1 w_{max,b} - w_1)(w_2 - 1) - R_2 (R_2 w_{max,b} - w_2)(w_1 - 1))\, a11 \Big/ ($$

$$R_1 R_2 ((w_{max,b} - w_1)(R_2 - 1) - (w_{max,b} - w_2)(R_1 - 1)))$$

$$- \frac{(R_1 w_2 - R_2 w_1)(w_{max,b} - 1)\, a11}{(w_{max,b} - w_1)(R_2 - 1) - (w_{max,b} - w_2)(R_1 - 1)} - a11^2 \Bigg) / a21$$

Specify $b_{2,2}$ as in Equation34:

$$> b22 := \frac{(R_1 w_{max,b} - w_1)(w_2 - 1)}{R_2\, c3} - \frac{(R_2 w_{max,b} - w_2)(w_1 - 1)}{R_1\, c3} + \frac{(R_2 w_1 - R_1 w_2)(w_{max,b} - 1)}{c3} - a11$$

$$b22 := \frac{(R_1 w_{max,b} - w_1)(w_2 - 1)}{R_2 ((w_{max,b} - w_1)(R_2 - 1) - (w_{max,b} - w_2)(R_1 - 1))} - \frac{(R_2 w_{max,b} - w_2)(w_1 - 1)}{R_1 ((w_{max,b} - w_1)(R_2 - 1) - (w_{max,b} - w_2)(R_1 - 1))} + \frac{(R_2 w_1 - R_1 w_2)(w_{max,b} - 1)}{(w_{max,b} - w_1)(R_2 - 1) - (w_{max,b} - w_2)(R_1 - 1)} - a11$$

Now define the wage-profits curve for the specified coefficients of production:

$$> w := R \to \frac{(a11\, b22 - b12\, a21) R^2 - (a11 + b22) R + 1}{(a21\, b02 - b22\, a01) R + a01}$$

$$w := R \to \frac{(a11\, b22 - b12\, a21) R^2 - (a11 + b22) R + 1}{(a21\, b02 - b22\, a01) R + a01}$$

Show that the maximum wage is $w_{max,b}$ and that the wage-profits curve goes through the desired switch points:

> simplify(w(1))

$$w_{max,b}$$

> simplify(w(R_1))

$$w_1$$

> simplify(w(R_2))

$$w_2$$

Now check Equation 35. The left hand side is:

$$> f := (1 - a11)(1 - b22) - b12\, a21$$

$$f := (1 - a11) \Bigg(1 - \frac{(R_1 w_{max,b} - w_1)(w_2 - 1)}{R_2 ((w_{max,b} - w_1)(R_2 - 1) - (w_{max,b} - w_2)(R_1 - 1))} + \frac{(R_2 w_{max,b} - w_2)(w_1 - 1)}{R_1 ((w_{max,b} - w_1)(R_2 - 1) - (w_{max,b} - w_2)(R_1 - 1))} - \frac{(R_2 w_1 - R_1 w_2)(w_{max,b} - 1)}{(w_{max,b} - w_1)(R_2 - 1) - (w_{max,b} - w_2)(R_1 - 1)} + a11\Bigg)$$

$$- \frac{R_2 (w_{max,b} - w_2)(w_1 - 1) - R_1 (w_{max,b} - w_1)(w_2 - 1)}{R_1 R_2 ((w_{max,b} - w_1)(R_2 - 1) - (w_{max,b} - w_2)(R_1 - 1))} + \frac{(w_{max,b} - 1)(w_1 - w_2)}{(w_{max,b} - w_1)(R_2 - 1) - (w_{max,b} - w_2)(R_1 - 1)} -$$

$$(R_1 (R_1 w_{max,b} - w_1)(w_2 - 1) - R_2 (R_2 w_{max,b} - w_2)(w_1 - 1))\, a11 \Big/ ($$

$$R_1 R_2 ((w_{max,b} - w_1)(R_2 - 1) - (w_{max,b} - w_2)(R_1 - 1))$$

$$+ \frac{(R_1 w_2 - R_2 w_1)(w_{max,b} - 1)\, a11}{(w_{max,b} - w_1)(R_2 - 1) - (w_{max,b} - w_2)(R_1 - 1)} + a11^2$$

The right hand side is:

$$> g := \frac{(c1 + (R_1 - 1)(R_2 - 1)(R_1 w_2 - R_2 w_1))\, w_{max,b}}{R_1 R_2\, c3}$$

$$g := ((R_2 - R_1)(R_1 - 1)(R_2 - 1) + (R_1 - 1)(R_2 - 1)(R_1 w_2 - R_2 w_1))$$

$$w_{max,b} \Big/ (R_1 R_2 ((w_{max,b} - w_1)(R_2 - 1) - (w_{max,b} - w_2)(R_1 - 1)))$$

And these sides are equal:

> simplify(f − g)

$$0$$

> *restart*

6.0 A Demonstration

Here the example is derived. Define the parameters from Table 1:

$$> R_1 := \frac{1}{2} + 1$$

$$R_1 := \frac{3}{2}$$

$$> w_1 := \frac{327}{500}$$

$$w_1 := \frac{327}{500}$$

$$> R_2 := \frac{3}{2} + 1$$

$$R_2 := \frac{5}{2}$$

$$> w_2 := \frac{67}{250}$$

$$w_2 := \frac{67}{250}$$

$$> w_{max} := \frac{4}{5}$$

$$w_{max} := \frac{4}{5}$$

$$> w_{max,b} := \frac{21}{25}$$

$$w_{max,b} := \frac{21}{25}$$

$$> a_{2,2} := \frac{1}{100}$$

$$a_{2,2} := \frac{1}{100}$$

Now find the remaining coefficients of production for the first technique. Use the equations in Section 3.

$$> c_1 := (R_2 - R_1)(R_1 - 1)(R_2 - 1)$$

$$c_1 := \frac{3}{4}$$

$$> c_2 := (w_{max} - w_1)(R_2 - 1) - (w_{max} - w_2)(R_1 - 1)$$

$$c_2 := \frac{-47}{1000}$$

$$> a_{1,1} := \frac{(R_1 w_{max} - w_1)(w_2 - 1)}{c_2 R_2} - \frac{(R_2 w_{max} - w_2)(w_1 - 1)}{c_2 R_1} + \frac{(R_2 w_1 - R_1 w_2)(w_{max} - 1)}{c_2} - a_{2,2}$$

$$a_{1,1} := \frac{48631}{352500}$$

$$> a_{1,2} := (R_2 (w_1 - 1)(w_{max} - w_2) - R_1 (w_2 - 1)(w_{max} - w_1) - R_1 R_2 (w_{max} - 1)(w_1 - w_2) + a_{2,2} R_2 (R_1 R_2 (w_{max} - 1) w_1 - (R_2 w_{max} - w_2)(w_1 - 1)) - a_{2,2} R_1 (R_1 R_2 (w_{max} - 1) w_2 - (R_1 w_{max} - w_1)(w_2 - 1)) - a_{2,2}{}^2 c_2 R_1 R_2) \big/ (R_2 ((R_2 - 1) w_1 - R_1 R_2 w_{max}) - R_1 ((R_1 - 1) w_2 - R_1 R_2 w_{max}) + c_1 + c_2 R_1 R_2 a_{2,2})$$

$$a_{1,2} := \frac{707677}{17500}$$

$$> a_{1,2} - 40$$

$$\frac{7677}{17500}$$

$$> a_{2,1} := a_{2,2} - \frac{(R_1 - 1) w_2}{c_2 R_2} + \frac{(R_2 - 1) w_1}{c_2 R_1} - \frac{(R_2 - R_1) w_{max}}{c_2} + \frac{c_1}{c_2 R_1 R_2}$$

$$a_{2,1} := \frac{7}{4700}$$

Find the coefficients of production for the second process in the first industry. Use the equations in Section 4.0.

$$> c_3 := (w_{max,b} - w_1)(R_2 - 1) - (w_{max,b} - w_2)(R_1 - 1)$$

$$c_3 := \frac{-7}{1000}$$

$$> c_4 := -a_{1,2} a_{2,2} c_3 R_1 R_2 + a_{1,2} (R_2 (R_2 - 1)(w_{max,b} R_1 - w_1) - R_1 (R_1 - 1)(w_{max,b} R_2 - w_2)) + a_{2,2} (w_1 (R_1 - 1)(w_{max,b} R_2{}^2 - w_2) - w_2 (R_2 - 1)(w_{max,b} R_1{}^2 - w_1)) + w_2 (R_2 - 1)(w_{max,b} R_1 - w_1) - w_1 (R_1 - 1)(w_{max,b} R_2 - w_2)$$

$$c_4 := \frac{10080180513}{280000000}$$

$$> b01 := (a_{2,2}{}^2 c_3 R_1 R_2 + a_{2,2} ((R_1 - 1)(R_1 + 1)(w_{max,b} - R_2 w_2) - (R_2 - 1)(R_2 + 1)(w_{max,b} - R_1 w_1)) + a_{1,2} c_1 + (R_2 - 1)(w_{max,b} - R_1 w_1) - (R_1 - 1)(w_{max,b} - R_2 w_2)) \big/ c_4$$

$$b01 := \frac{1094855}{1312011}$$

$$> b_{1,1} := (a_{1,2} a_{2,2}{}^2 c_3 R_1 R_2 + a_{1,2} a_{2,2} R_1 (R_1 (R_2 + 1)(w_{max,b} - w_2) - (w_1 - w_2))$$

$$-a_{1,2}\,a_{2,2}\,R_2\,(R_2\,(R_1+1)\,(w_{max,b}-w_1)+w_1-w_2)+$$
$$a_{1,2}\,((R_2+1)\,(R_2-1)\,(w_{max,b}-w_1)-(R_1+1)\,(R_1-1)\,(w_{max,b}-w_2))$$
$$+a_{2,2}\,(w_1\,(R_1-1)\,(R_2\,w_{max,b}-w_2)-w_2\,(R_2-1)\,(R_1\,w_{max,b}-w_1))$$
$$+w_2\,(R_2-1)\,(w_{max,b}-w_1)-w_1\,(R_1-1)\,(w_{max,b}-w_2))\Big/c_4$$

$$b_{1,1}:=\frac{191109761}{656005500}$$

> $b_{2,1}:=(1-a_{2,2})\,(1-R_1\,a_{2,2})\,(1-R_2\,a_{2,2})$
$((R_1-1)\,(w_{max,b}-w_2)-(R_2-1)\,(w_{max,b}-w_1))\Big/c_4$

$$b_{2,1}:=\frac{539}{2915580}$$

Now we check what we have so far.

> *restart*

Reset the coefficients of production for the two techniques.

> $a:=\langle\langle\frac{48631}{352500},\frac{7}{4700}\rangle\,|\,\langle\frac{707677}{17500},\frac{1}{100}\rangle\rangle$

$$a:=\begin{bmatrix}\frac{48631}{352500} & \frac{707677}{17500}\\ \frac{7}{4700} & \frac{1}{100}\end{bmatrix}$$

> $b01:=\frac{1094855}{1312011}$

$$b01:=\frac{1094855}{1312011}$$

> $b:=\langle\langle\frac{191109761}{656005500},\frac{539}{2915580}\rangle\,|\,\langle\frac{707677}{17500},\frac{1}{100}\rangle\rangle$

$$b:=\begin{bmatrix}\frac{191109761}{656005500} & \frac{707677}{17500}\\ \frac{539}{2915580} & \frac{1}{100}\end{bmatrix}$$

Specify the wage-profits curves for the two techniques.

> $walpha:=R\rightarrow\frac{(a_{1,1}\,a_{2,2}-a_{1,2}\,a_{2,1})\,R^2-(a_{1,1}+a_{2,2})\,R+1}{(a_{2,1}-a_{2,2})\,R+1}$

$$walpha:=R\rightarrow\frac{(a_{1,1}\,a_{2,2}-a_{1,2}\,a_{2,1})\,R^2-(a_{1,1}+a_{2,2})\,R+1}{(a_{2,1}-a_{2,2})\,R+1}$$

> $wbeta:=R\rightarrow\frac{(b_{1,1}\,b_{2,2}-b_{1,2}\,b_{2,1})\,R^2-(b_{1,1}+b_{2,2})\,R+1}{(b_{2,1}-b_{2,2}\,b01)\,R+b01}$

$$wbeta:=R\rightarrow\frac{(b_{1,1}\,b_{2,2}-b_{1,2}\,b_{2,1})\,R^2-(b_{1,1}+b_{2,2})\,R+1}{(b_{2,1}-b_{2,2}\,b01)\,R+b01}$$

Express the wage-profits curves in closed form:

> simplify(walpha(R))

$$\frac{1}{375}\,\frac{5186\,R^2+13039\,R-88125}{2\,R-235}$$

> simplify(wbeta(R))

$$\frac{1}{125}\frac{748276 R^2 + 49417454 R - 164001375}{10706 R - 1094855}$$

Switch points are among the roots of

walpha(*R*) = wbeta(*R*)

The roots of this equation are the zeros of the following polynomial:

> $f := R \rightarrow 375 (2 R - 235) (748276 R^2 + 49417454 R - 164001375) - 125 (10706 R - 1094855) (5186 R^2 + 13039 R - 88125)$

$$f := R \rightarrow 375 (2 R - 235) (748276 R^2 + 49417454 R - 164001375) - 125 (10706 R - 1094855) (5186 R^2 + 13039 R - 88125)$$

> simplify(f(*R*))

$$-6378957500 R^3 + 663411580000 R^2 - 2575504090625 R + 2392109062500$$

The zeros of the above polynomial are the same as the zeros to the following polynomial:

> $g := R \rightarrow \frac{f(R)}{-1594739375}$

$$g := R \rightarrow -\frac{1}{1594739375} f(R)$$

> simplify(g(*R*))

$$4 R^3 - 416 R^2 + 1615 R - 1500$$

These zeros are:

> solve(g(*R*) = 0, *R*)

$$\frac{3}{2}, \frac{5}{2}, 100$$

Now echo out the maximum wages and the wage at the rates of profits found above, for the two wage-profits curves:

> walpha(1)

$$\frac{4}{5}$$

> wbeta(1)

$$\frac{21}{25}$$

> $\text{walpha}\left(\frac{3}{2}\right)$

$$\frac{327}{500}$$

> $\text{wbeta}\left(\frac{3}{2}\right)$

$$\frac{327}{500}$$

> $\text{walpha}\left(\frac{5}{2}\right)$

$$\frac{67}{250}$$

> $\text{wbeta}\left(\frac{5}{2}\right)$

$$\frac{67}{250}$$

To be economically meaningful, the wage must be non-negative at a switch point. So the following is not a switch point:

```
> walpha(100)
```

$$\frac{-707677}{175}$$

```
> wbeta(100)
```

$$\frac{-707677}{175}$$

```
> ?
> ?
> ?
```