Detailed Working Through Garegnani Reswitching Example

Robert L. Vienneau

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1.0 Introduction

This document merely steps one through the example in the Appendix, part I, in Garegnani 1970. Garegnani does not include intertemporal utility-maximizing in his model, which is fine for his point. The model below has a log-linear separable utility function in a representative agent, overlapping generations approach, as in Ferretti 2004. The utility function here is simpler than Ferretti in that the agents are constrained to sell their labor-power in the first year and to be retired the remaining years of their life; Ferretti includes arguments for leisure in each year. On the other hand, Ferretti assumes each generation lives for three years; I assume they live for four.

2.0 Technology

Assume a simple economy in which gross outputs consist only of u-grade steel and corn. Steel can be produced in a continuum of grades, each grade denoted by the index u, $u \geq 0$. Steel is strictly a capital good, while corn is used strictly for consumption. U-grade steel is produced from inputs of $a_{0,1}^{u}$ person-years of labor and $A_{u,1,1}$ tons of u-grade steel per ton output. Corn is produced from inputs of $a_{0,2}^{u}$ person-years of labor and $A_{u,1,2}$ tons of u-grade steel per corn output. In both cases, Constant Returns to Scale (CRS) are assumed. This is a circulating capital model; all steel used in production is totally used up in production in a single production cycle. Garegnani deliberately adopts the assumptions of Samuelson 1962, generalized to allow physical capital-intensity to vary across sectors. Anyways, Garegnani assumes the coefficients of production specified in Displays 1 and 2:

$$\mathbf{a}_0^u = \left[1 \quad \frac{30 + 11u^{\left(\frac{11}{10}\right)} + u^{\left(\frac{11}{5}\right)} - 27e^{-2u}}{6 + u^{\left(\frac{11}{10}\right)}} \right] \tag{1}$$

$$\mathbf{A}_u = \begin{bmatrix} \dfrac{5 + u^{\left(\frac{11}{10}\right)}}{6 + u^{\left(\frac{11}{10}\right)}} & \dfrac{27e^{-2u}}{\left\{6 + u^{\left(\frac{11}{10}\right)}\right\}^2} \\ 0 & 0 \end{bmatrix} \tag{2}$$

3.0 Quantity Flows

Table 1 shows how much u-grade steel and corn are produced per worker in a stationary state. Note that the inputs of labor in this table add up to unity and that the sum of the steel inputs is equal to the quantity of steel produced. The ratio of inputs to output in each column is as specified by the coefficients of production. The amount of u-grade steel used per worker and the net output of corn per worker are also shown.

Table 1: Stationary State Quantity Flows With u-Grade Steel

Inputs	**Steel Industry**	**Corn Industry**
Labor	$\dfrac{27e^{-2u}}{30+11u^{\left(\frac{11}{10}\right)}+u^{\left(\frac{11}{5}\right)}}$ Yrs	$\dfrac{30+11u^{\left(\frac{11}{10}\right)}+u^{\left(\frac{11}{5}\right)}-27e^{-2u}}{\left\{6+u^{\left(\frac{11}{10}\right)}\right\}\left\{5+u^{\left(\frac{11}{10}\right)}\right\}}$ Yrs.
Steel	$\dfrac{27e^{-2u}}{\left\{6+u^{\left(\frac{11}{10}\right)}\right\}^2}$ Tons	$\dfrac{27e^{-2u}}{\left\{6+u^{\left(\frac{11}{10}\right)}\right\}^2\left\{5+u^{\left(\frac{11}{10}\right)}\right\}}$ Tons
Corn	0 Bushels	0 Bushels
Outputs	$\dfrac{27e^{-2u}}{30+11u^{\left(\frac{11}{10}\right)}+u^{\left(\frac{11}{5}\right)}}$ Tons	$\dfrac{1}{5+u^{\left(\frac{11}{10}\right)}}$ Bushels

Capital Per Worker: $\dfrac{27e^{-2u}}{30+11u^{\left(\frac{11}{10}\right)}+u^{\left(\frac{11}{5}\right)}}$ Tons

Net Output Per Worker: $\dfrac{1}{5+u^{\left(\frac{11}{10}\right)}}$ Bushels

4.0 Price Equations

If stationary state prices prevail, and the rate of profits is the same in both sectors, Equation 3 is satisfied:

$$\left[p(r,u) \quad 1\right]\mathbf{A}_u(1+r)+\mathbf{a}_0^u w(r,u)=\left[p(r,u) \quad 1\right] \tag{3}$$

I have assumed that the laborers working through the year are paid out of the harvest at the end of the year. Equation 3, given the coefficients of production, is a system of two equations in three unknowns. Equations 4 and 5 give the solution for the wage and the price of u-grade steel in terms of the rate of profits:

$$w(r,u) = \frac{u^{\left(\frac{11}{10}\right)} r + 5r - 1}{u^{\left(\frac{11}{5}\right)} r + 10u^{\left(\frac{11}{10}\right)} r - u^{\left(\frac{11}{10}\right)} + 25r - 5 - 27re^{-2u}} \tag{4}$$

$$p(r,u) = \frac{-6 - u^{\left(\frac{11}{10}\right)}}{u^{\left(\frac{11}{5}\right)} r + 10u^{\left(\frac{11}{10}\right)} r - u^{\left(\frac{11}{10}\right)} + 25r - 5 - 27re^{-2u}} \tag{5}$$

Figure 1 graphs Equation 4 for four values of *u*. The cost-minimizing outer envelope wage-profits frontier is also shown. The wage-profits curve for *u* = 0 is of a different convexity than the other three wage-profits curves shown.

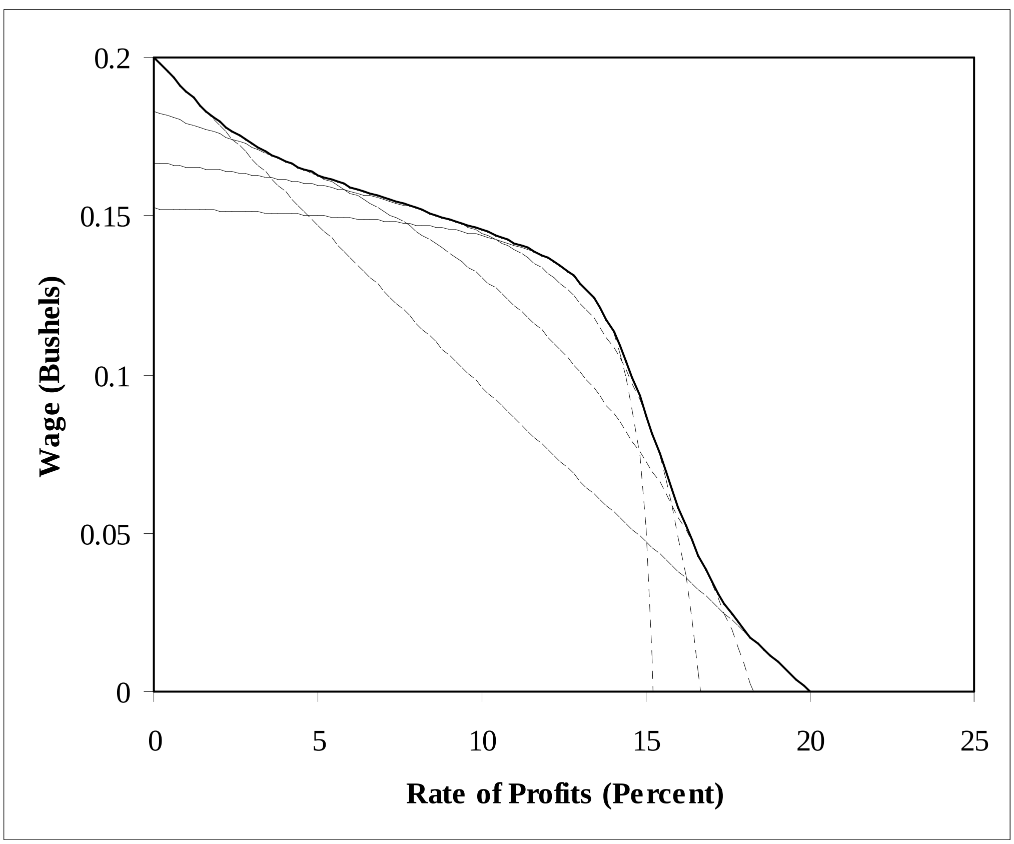

Figure 1: Wage-Profits Frontier

Competitive firms select the grade of steel to use to minimize costs, as is shown by the wage-profits frontier. The grade of steel varies continuously along the frontier, with each cost-minimizing grade of steel used for a low and high rate of profits, except for $u = u^* \approx 1.51$. In other words, this is a continuous reswitching example. Note that since only one value of *u* is cost minimizing for each rate of profits, all points on the wage-profits frontier are non-switching points. The cost-minimizing grade of steel is the value of *u* that results in equating the derivative of $w(r, u)$ with respect to *u* to zero. This equation was solved numerically, by Newton's method. A table in the appendix displays the cost-minimizing grade of steel for a wide range of rates of profit. Once one has found the grade of steel as a function of the rate of profits, one can produce a number of interesting graphs. Figure 2 shows the net output per worker as a function of the rate of profits. Figure 3 shows how the wage and the cost minimizing labor intensity relate. Figure 4 shows the value of gross investment per worker; the savings functions are explained in Section 5 below. Since the value of capital per worker and net output per worker are both found as functions of the rate of profits, the aggregate production "function" can be plotted parametrically. As shown in Figure 5, the production "function" is, amazingly enough, a smooth loop in this example.

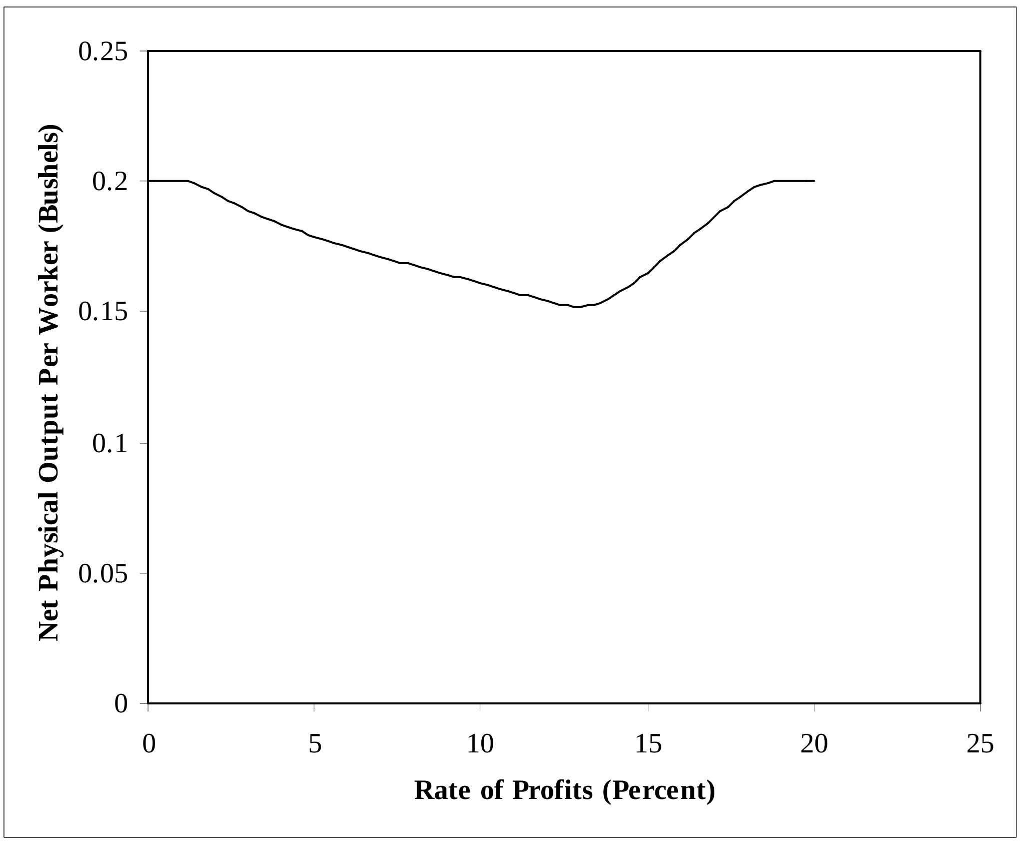

Figure 2: Net Output

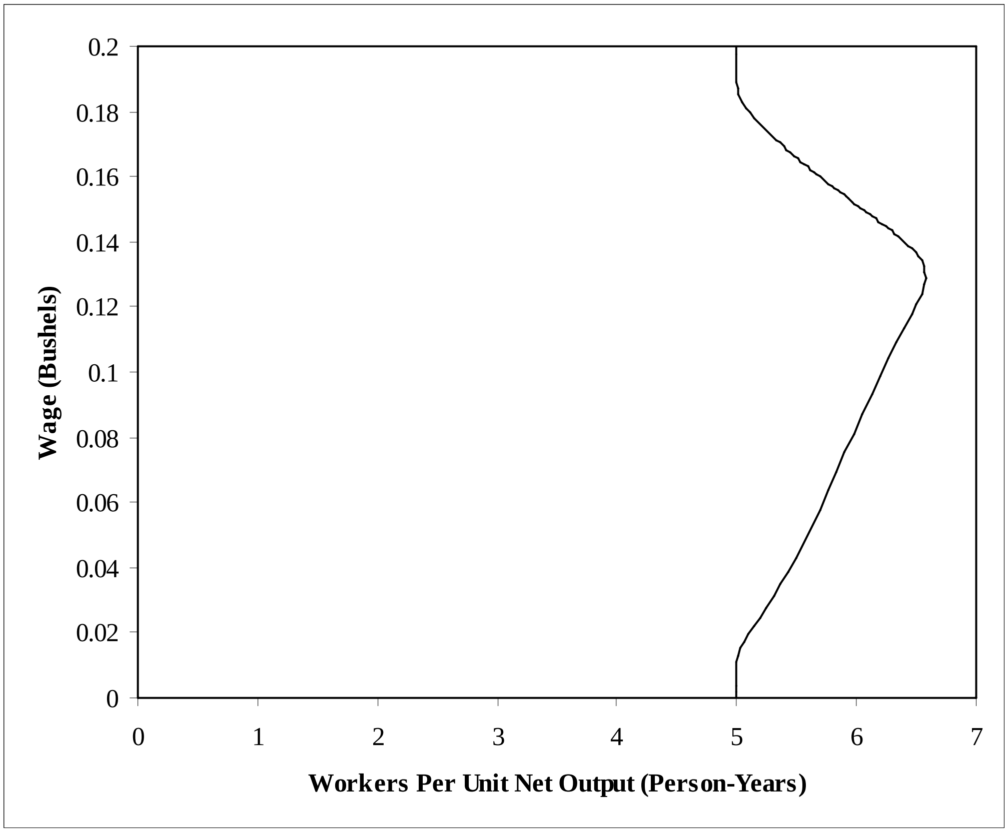

Figure 3: Labor Per Unit Net Output

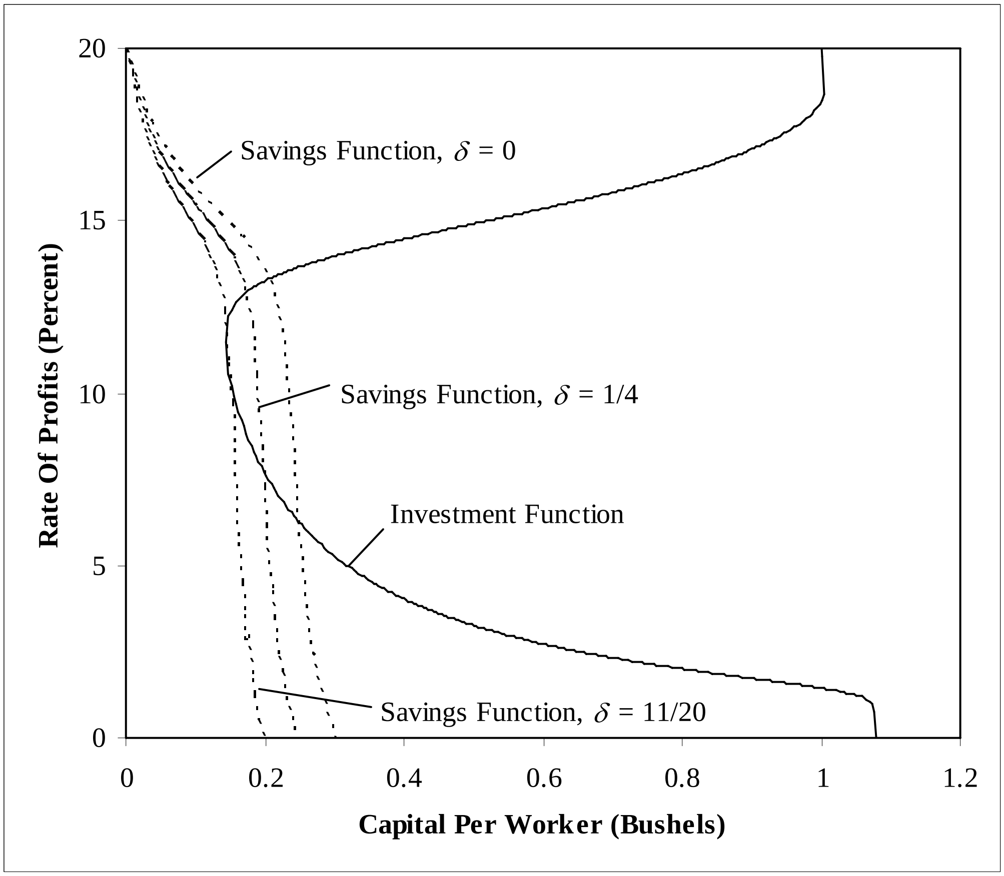

Figure 4: Investment and Savings

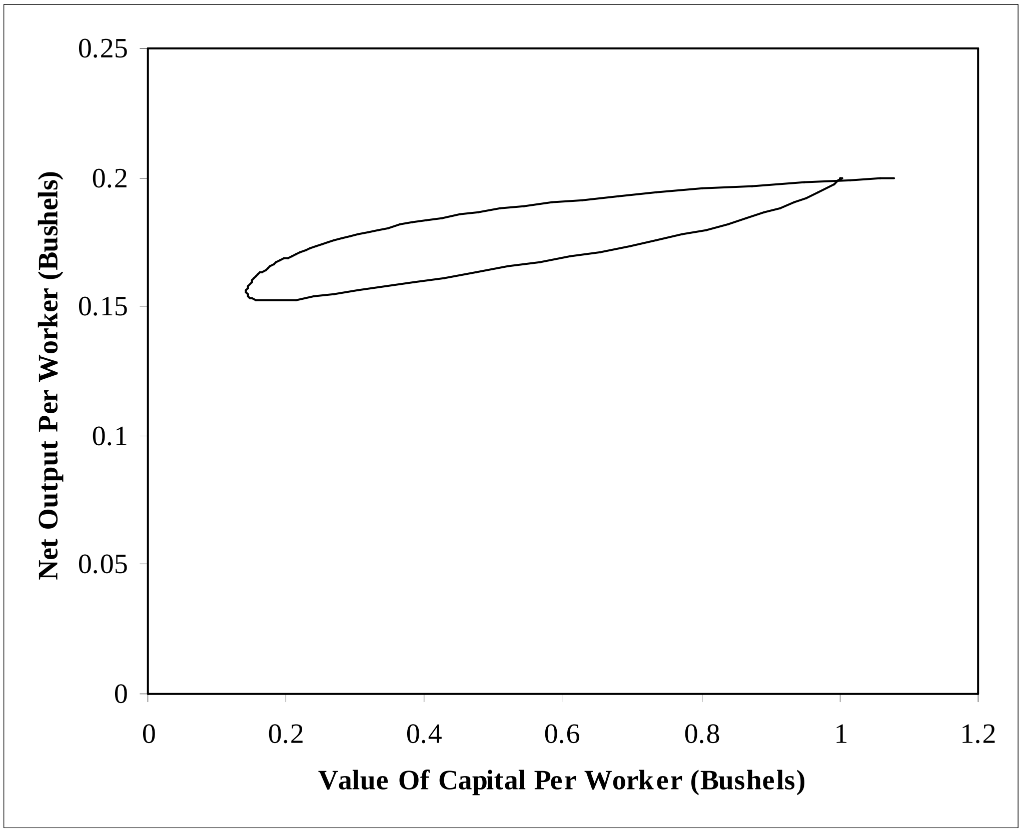

Figure 5: Pseudo-Production "Function"

5.0 Utility Maximizing

One way to close the model and define a full equilibrium of the economy is to introduce utility-maximization within an overlapping generations model. Assume each worker in the overlapping generations model lives four years. The worker is born at the beginning of a year, sells a person-year of labor services for use during the first year of his life, purchases some corn to consume immediately out of the wages paid at the end of the year, and saves the remainder of his wages to be used to purchase corn for consumption at the end of each of the last three years of his life. Even further, assume a single worker is born each year. Assume each worker has an identical log-linear separable utility function. Each worker faces the constraint that his immediate consumption and the present value of his retirement consumption add up to the wage in the year that he works. In other words each worker solves the constrained utility-maximization problem in Display 6:

$$\text{Maximize } U(c_0, c_1, c_2, c_3) = \sum_{i=0}^{3} \frac{1}{(1+\delta)^i} \ln(c_i) \qquad (6)$$

$$\text{Such that } \sum_{i=0}^{3} \frac{c_i}{(1+r)^i} = w$$

As δ becomes larger, the agent becomes more impatient. When δ is zero, the agent weighs equally the utility of consumption at the end of each time period. Display 6 gives the marginal conditions:

$$\frac{\dfrac{\partial U}{\partial c_{i-1}}}{\dfrac{\partial U}{\partial c_i}} = \frac{(1+\delta)c_i}{c_{i-1}} = 1+r, \quad i = 1, 2, 3 \tag{7}$$

The marginal conditions allow one to express consumption in future years in terms of the consumption in the first year:

$$c_i = \left(\frac{1+r}{1+\delta}\right)^i c_0, \quad i = 0, 1, 2, 3 \tag{8}$$

Substituting consumption in future years into the budget constraint, one obtains 4:

$$c_0 = \frac{1}{\sum_{i=0}^{3} \frac{1}{(1+\delta)^i}} w = \frac{w}{1 + \frac{1}{1+\delta} + \frac{1}{(1+\delta)^2} + \frac{1}{(1+\delta)^3}} = \frac{\delta(1+\delta)^3}{(1+\delta)^4 - 1} w \tag{9}$$

Hence, consumption at the end of each year in the agent's life is:

$$c_i = \frac{\delta(1+\delta)^{3-i}}{(1+\delta)^4 - 1} (1+r)^i w, \quad i = 0, 1, 2, 3 \tag{10}$$

Stationary state savings is found by adding up the savings of the different agents alive at the end of a given year:

$$S(r) = \left[\frac{c_1}{1+r} + \frac{c_2}{(1+r)^2} + \frac{c_3}{(1+r)^3}\right] + \left[\frac{c_2}{1+r} + \frac{c_3}{(1+r)^2}\right] + \frac{c_3}{1+r} \tag{11}$$

Or:

$$S(r) = \frac{6 + 4\delta + \delta^2 + 4r + \delta r + r^2}{4 + 6\delta + 4\delta^2 + \delta^3} w(r) \tag{12}$$

Figure 4 shows savings functions for three different values of the parameter of the utility function.

The economy is in long period equilibrium when the value of stationary-state investment and savings are equal:

$$I(r) = S(r) \quad (13)$$

This a stock equilibrium condition; that is, the agents must collectively be willing to hold the capital goods used in a stationary state. Economy wide equilibria are shown in Figure 4 by intersections of the savings and investment functions. One can find equilibrium rates of profit for each value of δ in a certain interval. I solved for the rate of profits numerically. Values of the rate of profits were found around each intersection for each of a set of given values of δ. These values were used in a linear interpolation to find the equilibrium rate of profits. The numerical solutions are tabulated in the appendix. Figure 6 graphs the equilibrium rates of profits as functions of the utility function parameter.

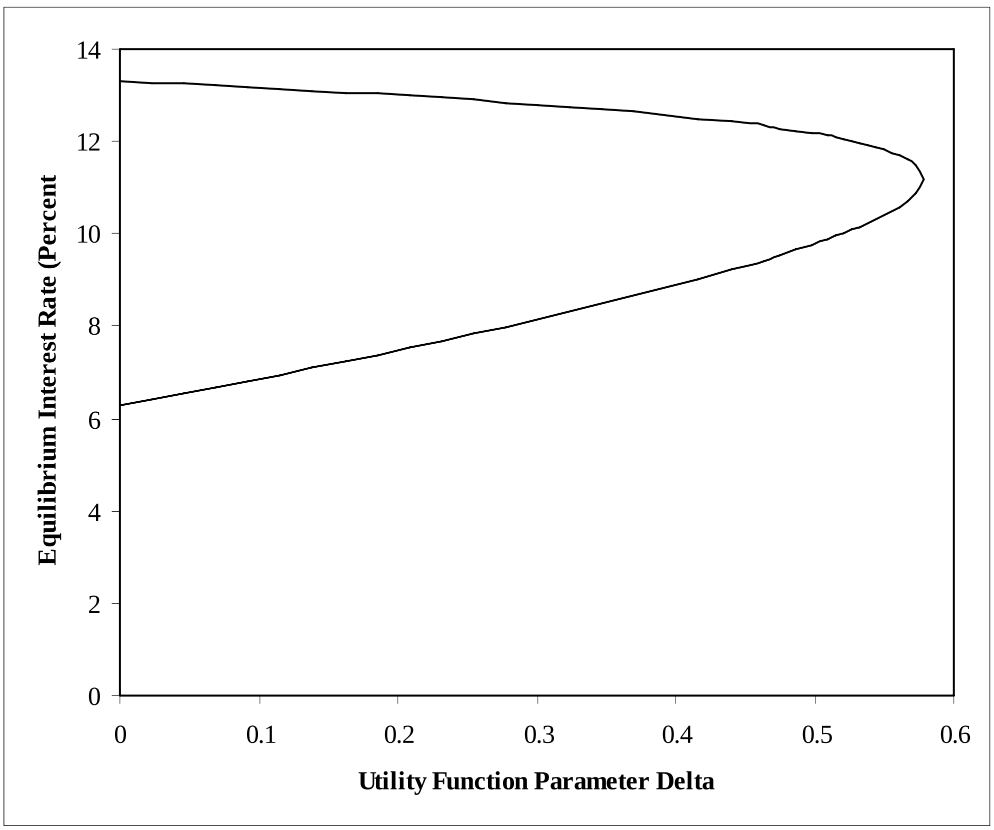

Figure 6: Equilibria of the Economy

As δ falls in the Figure 6, moving from right to left, the original single equilibrium bifurcates into two equilibria. The agents become more patient, and on the upper curve, a willingness to supply more "capital" is manifested in a *higher* equilibrium interest rate[1]. So much for explaining the rate of interest in long period models by the interaction of well-behaved supply and demand functions in the "capital" market. It

[1] The dynamic language is misleading. Assertions are being made only about comparisons of long period equilibria. The analysis must be extended to consider dynamics, whether in logical or Joan Robinson's "historical" time. See Robinson 1974 and 1975.

follows that explaining wages and employment in long period models by the interaction of well-behaved supply and demand functions is just as unjustified.

Appendix

Table A-1: Steel Grade As Function Of Rate Of Profits

***r* (Percent)**	***u***	***r* (Percent)**	***u***	***r* (Percent)**	***u***
0	0	6.8	0.83228639	13.8	1.40874362
0.2	9.19349E-11	7	0.85480649	14	1.358415423
0.4	1.03847E-07	7.2	0.87719637	14.2	1.302032992
0.6	6.59576E-06	7.4	0.899484791	14.4	1.241409257
0.8	0.000128514	7.6	0.921699787	14.6	1.177960852
1	0.001284423	7.8	0.943868838	14.8	1.112747598
1.2	0.007679347	8	0.966019019	15	1.04654667
1.4	0.026866118	8.2	0.988177139	15.2	0.979923797
1.6	0.059120564	8.4	1.010369851	15.4	0.91329059
1.8	0.097914596	8.6	1.032623747	15.6	0.846947518
2	0.138545255	8.8	1.054965412	15.8	0.781115424
2.2	0.178885204	9	1.077421445	16	0.715958592
2.4	0.218089803	9.2	1.10001841	16.2	0.651601996
2.6	0.255868539	9.4	1.122782714	16.4	0.588144805
2.8	0.292169017	9.6	1.145740349	16.6	0.525671837
3	0.327040919	9.8	1.168916463	16.8	0.464264594
3.2	0.360575467	10	1.192334653	17	0.404013795
3.4	0.392877717	10.2	1.216015863	17.2	0.345036265
3.6	0.424053781	10.4	1.23997667	17.4	0.287501279
3.8	0.454205091	10.6	1.26422664	17.6	0.231676533
4	0.483426074	10.8	1.288764269	17.8	0.178015661
4.2	0.511803488	11	1.313570704	18	0.127336404
4.4	0.539416562	11.2	1.33860002	18.2	0.081193294
4.6	0.566337493	11.4	1.363764082	18.4	0.042558748
4.8	0.592632095	11.6	1.388908913	18.6	0.016058514
5	0.61836048	11.8	1.413777931	18.8	0.003886755
5.2	0.643577708	12	1.437955562	19	0.000606249
5.4	0.668334399	12.2	1.460783973	19.2	5.97743E-05
5.6	0.692677285	12.4	1.481250244	19.4	3.05985E-06
5.8	0.716649707	12.6	1.497862303	19.6	4.81324E-08
6	0.740292058	12.8	1.508586252	19.8	4.25903E-11
6.2	0.763642179	13	1.511000687	20	0
6.4	0.786735707	13.2	1.502829025		
6.6	0.809606394	13.4	1.482740647		

Table A-2: Lower Equilibrium

δ	r_1 (Percent)	$I(r_1)$-$S(r_1)$	r_1 (Percent)	$I(r_1)$-$S(r_1)$	r (Percent)
0	6.2	0.00352412	6.4	-0.0042066	6.29117184
0.02314	6.4	0.00059886	6.6	-0.0066086	6.41661778
0.04628	6.4	0.00529013	6.6	-0.001929	6.54655866
0.06942	6.6	0.00263784	6.8	-0.0041013	6.67828417
0.09256	6.8	0.00034329	7	-0.0059545	6.81090192
0.1157	6.8	0.00467837	7	-0.0016302	6.94831792
0.13884	7	0.00258686	7.2	-0.0033139	7.08767888
0.16198	7.2	0.00078738	7.4	-0.0047355	7.22851338
0.18512	7.2	0.00478537	7.4	-0.0007476	7.37297654
0.20826	7.4	0.00313939	7.6	-0.0020419	7.52118179
0.2314	7.6	0.00173658	7.8	-0.0031163	7.67156905
0.25454	7.8	0.00055626	8	-0.0039885	7.82447918
0.27768	7.8	0.00413468	8	-0.0004194	7.98158135
0.30082	8	0.00305811	8.2	-0.0012055	8.14345168
0.32396	8.2	0.00217359	8.4	-0.0018149	8.30899313
0.3471	8.4	0.00146835	8.6	-0.002258	8.47880902
0.37024	8.6	0.00093186	8.8	-0.0025432	8.6536313
0.39338	8.8	0.00055579	9	-0.0026766	8.8343888
0.41652	9	0.00033394	9.2	-0.0026621	9.02229209
0.43966	9.2	0.0002623	9.4	-0.0025014	9.2189818
0.4628	9.4	0.00033913	9.6	-0.0021938	9.42677768
0.47437	9.4	0.00173189	9.6	-0.0008054	9.53651494
0.48594	9.6	0.00056513	9.8	-0.0017357	9.64912401
0.49751	9.6	0.00191793	9.8	-0.0003873	9.76639815
0.50908	9.8	0.00094378	10	-0.0011204	9.89144358
0.514865	9.8	0.00160286	10	-0.0004636	9.955131
0.52065	10	0.00018906	10.2	-0.0016256	10.020837
0.526435	10	0.00083748	10.2	-0.0009794	10.0921888
0.53222	10	0.00148171	10.2	-0.0003375	10.162896
0.538005	10.2	0.00030038	10.4	-0.0012537	10.238657
0.54379	10.2	0.00093411	10.4	-0.0006223	10.3200339
0.549575	10.4	5.0852E-06	10.6	-0.0012666	10.4007998
0.55536	10.4	0.00062839	10.6	-0.0006457	10.4986414
0.561145	10.4	0.00124769	10.6	-0.00002883	10.595483
0.56693	10.6	0.00058411	10.8	-0.0003805	10.721108
0.572715	10.8	0.00022598	11	-0.000389	10.8734918
0.5785	11	0.00021103	11.2	-0.000002355	11.1977927
0.584285					

Table A-3: Upper Equilibrium

δ	r_1 (Percent)	$I(r_1)$-$S(r_1)$	r_1 (Percent)	$I(r_1)$-$S(r_1)$	r (Percent)
0	6.2	0.00352412	6.4	-0.0042066	13.3132085
0.02314	6.4	0.00059886	6.6	-0.0066086	13.2799893
0.04628	6.4	0.00529013	6.6	-0.001929	13.2473753
0.06942	6.6	0.00263784	6.8	-0.0041013	13.2153701
0.09256	6.8	0.00034329	7	-0.0059545	13.1795425
0.1157	6.8	0.00467837	7	-0.0016302	13.1402301
0.13884	7	0.00258686	7.2	-0.0033139	13.1016764
0.16198	7.2	0.00078738	7.4	-0.0047355	13.0638782
0.18512	7.2	0.00478537	7.4	-0.0007476	13.0268298
0.20826	7.4	0.00313939	7.6	-0.0020419	12.9875059
0.2314	7.6	0.00173658	7.8	-0.0031163	12.9405786
0.25454	7.8	0.00055626	8	-0.0039885	12.8961096
0.27768	7.8	0.00413468	8	-0.0004194	12.8495213
0.30082	8	0.00305811	8.2	-0.0012055	12.8053702
0.32396	8.2	0.00217359	8.4	-0.0018149	12.7487823
0.3471	8.4	0.00146835	8.6	-0.002258	12.6914617
0.37024	8.6	0.00093186	8.8	-0.0025432	12.6352819
0.39338	8.8	0.00055579	9	-0.0026766	12.5728011
0.41652	9	0.00033394	9.2	-0.0026621	12.4985367
0.43966	9.2	0.0002623	9.4	-0.0025014	12.4256947
0.4628	9.4	0.00033913	9.6	-0.0021938	12.3361028
0.47437	9.4	0.00173189	9.6	-0.0008054	12.286893
0.48594	9.6	0.00056513	9.8	-0.0017357	12.2394413
0.49751	9.6	0.00191793	9.8	-0.0003873	12.1854688
0.50908	9.8	0.00094378	10	-0.0011204	12.1172157
0.514865	9.8	0.00160286	10	-0.0004636	12.0832942
0.52065	10	0.00018906	10.2	-0.0016256	12.0495102
0.526435	10	0.00083748	10.2	-0.0009794	12.0158609
0.53222	10	0.00148171	10.2	-0.0003375	11.9739641
0.538005	10.2	0.00030038	10.4	-0.0012537	11.9247033
0.54379	10.2	0.00093411	10.4	-0.0006223	11.8756048
0.549575	10.4	5.0852E-06	10.6	-0.0012666	11.8266649
0.55536	10.4	0.00062839	10.6	-0.0006457	11.7650589
0.561145	10.4	0.00124769	10.6	-0.00002883	11.6881702
0.56693	10.6	0.00058411	10.8	-0.0003805	11.6114208
0.572715	10.8	0.00022598	11	-0.000389	11.4795553
0.5785	11	0.00021103	11.2	-0.000002355	11.2017951
0.584285					

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