A Model For Exploring Manifestations of Capital-Theoretic 'Paradoxes' in Temporary Equilibria

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Abstract: In the Cambridge Capital Controversy, critics associated with Cambridge, UK, attack the logical coherence of neoclassical theory and claim to outline an alternative approach to economics. The most prominent neoclassical economists responding to the controversy acknowledge that many models in the mainstream literature are problematic in theory. Neoclassical economists argue, however, the canonical neoclassical theory is best expressed in disaggregated short period General Equilibrium models of intertemporal and temporary equilibria. They claim belief in the logical consistency and coherence of these models is and should be unaffected by Cambridge criticism. Recently, a dispute has arisen whether capital-theoretic paradoxes reappear in these short period models, in a different form. This paper presents a short period, overlapping-generations General Equilibrium model for exploring these issues. Further work is needed to analyze this model in light of the capital controversy.

Keywords: B51: Sraffian Economics, C61: Dynamic Analysis, C67: Input-Output Models, D33: Factor Income Distribution, D57: Input-Output Analysis, D90: Intertemporal Choice, E22: Capital Theory.

1.0 Introduction

The Cambridge Capital Controversy (CCC) is a long-lasting, complex, and multifaceted affair.[1] Critics associated with Cambridge, UK, attack the logical coherence of neoclassical theory and claim to outline an alternative approach to economics.[2] The Cambridge critics show examples of apparently paradoxical behavior, most notably reswitching and capital-reversing, within optimizing models. Neoclassical economists argue that the most rigorous versions of neoclassical theory, short run models of General Equilbrium theory, are unaffected by Cambridge criticism.

The objective of the research, whose start is described in this paper, is to provide an overview of these arguments and to provide a contribution to ongoing contemporary debates. In particular, this paper provides an overlapping generations, temporary equilibrium model in which to construct a traverse across a "perverse" switch point. As far as I am aware, no explicit numeric example of such a dynamic equilibrium path in

[1] Cohen and Harcourt (2003) provide a recent survey. Some letters in the Fall 2003 issue of the *Journal of Economic Perspectives* comment on that article.
[2] The constructive aspects of Sraffian economics fall outside the scope of this paper. In particular, neither the relationship between Sraffa's work and Classical and Marxian economics (see, for example, Garegnani 1984 and 1990a) nor the compatibility of Sraffian economics and the economics of Keynes are discussed here.

such a model has yet been described. Schefold (1997 and 2000) provides an example in an intertemporal equilibrium model with an agent with an infinite life.

The most prominent neoclassical economists responding to the CCC acknowledge that many models in the mainstream literature are problematic in theory. Austrian capital theory, Clarkian capital theory, and Solovian growth theory are among the approaches recognized to be theoretically questionable.[3] In a comparison of long period equilibrium positions, greater capital intensity need not be associated with a lower interest rate. Nor is greater sustainable consumption per head necessarily associated with a lower interest rate. The interest rate is generally not equal in equilibrium to the marginal product of capital[4]. Neoclassical economists[5] argue, however, the canonical neoclassical theory is best expressed in disaggregated short period General Equilibrium models of intertemporal[6] and temporary equilibria.[7] They claim belief in the logical consistency and coherence of these models is and should be unaffected by Cambridge criticism. In fact, inasmuch as reswitching and reverse capital deepening are analyzed within a neoclassical framework, these phenomena are implications of neoclassical theory.[8] Some of these economists suggest that Cambridge findings might have implications for stability analysis.[9, 10]

The turn towards short period General Equilibrium models has not gone unchallenged by partisans of Cambridge criticism. The critics have pointed out that the reliance on such short period models is a major change in economic method,[11] and have

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[3] Samuelson 1966 and 2001.

[4] Harcourt (1969). Burmeister (1980), for example, argues for measuring capital by Champernowne's chain index and thereby reasserts the equality of the interest rate and the marginal product of capital. Baldone (1984) is one response.

[5] Bliss 1975, Burmeister 1980, Dixit 1977, Hahn 1982, Samuelson 1975, and Stiglitz 1974.

[6] Debreu 1959.

[7] Burmeister 1980, Grandmont 1977, Hicks 1946.

[8] Mas-Colell 1989, Solow 1975.

[9] "In such instances of multiple rest points, the point toward which the optimizing economy converges depends crucially upon the initial capital-stock conditions. Therefore, the dynamics of the problem become very complex… Such conditions do not arise in regular economies [that is, economies without capital-reversing. The assumption of no capital-reversing] is sufficient for the uniqueness of the rest-point solution…" (Burmeister 1980).

[10] "Such a theory concerns an economy in full neoclassical equilibrium which, I have repeatedly argued, has nothing to fear from anything in Sraffa's or in his followers' work. But on the manner in which such an equilibrium is supposed to come about, neoclassical theory is highly unsatisfactory. Sraffa's work shows that certain simplified routes are very risky and not free from logical difficulties. The remarkable fact is that neither he nor the Sraffians have made anything of this" (Hahn 1982).

[11] For example, Garegnani (1976) and Milgate (1979).

questioned whether such models can tell us anything about actual economies.[12] In any approach to equilibrium in historical time, expectations and endowments of some capital goods would change; since expectations and endowments are among the data of such models, an equilibrium path in these models, as determined by the data, is of no relevance for the actual path of actual economies. In addition, economists sympathetic to Sraffianism[13] have argued that capital-theoretic paradoxes reappear in these short period models, but in a different form. Others[14] have disputed this claim. The research described in this paper makes explicit certain aspects of models of intertemporal and temporary equilibria on which there has been implicit agreement in these debates.[15, 16, 17, 18]

This paper does not go very far in investigating this controversy. Section 2 presents a short period, overlapping generations General Equilibrium model for exploring these issues. Section 3 defines stationary states in the model. Section 3 concludes with suggestions for further work, including the construction of dynamic equilibrium paths

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[12] For example, Dumenil and Levy (1985), Garegnani (1990b), and Petri (2004).

[13] Garegnani, P (2000, 2005a, and 2005b), Parrinello (2005), Rosser (1983 and 1991), and Schefold (1997, 2000, 2005a, and 2005b).

[14] Bloise and Richlin (2005), Ferretti (2004), and Mandler (1999, 2002, and 2005).

[15] “It is possible that the outputs produced in an Arrow-Debreu economy in the far distant future are independent of its initial endowments. That would mean that in such an economy the relative scarcities prevailing now would have no influence on the relative prices and rentals in the distant future. This should be enough to persuade the critics that the theory is not committed to a relative scarcity theory of distribution, though they seem to believe it is and that often motivates them in their attacks.” (Hahn 1981)

[16] “Even people who have made no study of economic theory are familiar with the idea that when something is more plentiful its price will be lower, and introductory courses on economic theory reinforce this common presumption with various examples. However, there is no support from the theory of general equilibrium for the proposition that an input to production will be cheaper in an economy where more of it is available. All that the theory declares is that the price of the use of an input which is more plentiful cannot be higher if all other inputs, all other outputs and all other input prices are in constant proportions to each other.” (Bliss 1975)

[17] “The Cambridge controversies demonstrated that – outside of one-commodity models, which all sides recognize as too restrictive and where, in any case, the labor substance theory of value is equally valid – there is no general neoclassical model where the scarcity theory of value can explain the set of all relative prices, including factor prices… …With the causal claims abandoned, what remains of neoclassical distribution theory are, using Samuelson’s terms, ‘parrot’-like specifications of simultaneous equation systems and (correct) statements about how factor returns are *equal to* or *measured by* disaggregated marginal productivities… Gone are the ‘useful’ claims about unambiguously signed changes in the interest rate resulting from changes in the quantity of capital.” (Cohen 1993, emphasis in the original)

[18] “Price as an index of scarcity cannot be made coherent in the supply-and-demand framework, which I think is the principal Sraffian critique.” (Harcourt, as quoted in King 1995).

with the numeric values given there. The list of symbols includes more symbols than appear in the models in this paper; some appear in this further work.

List of Symbols

$\Omega(t)$ A two-element column vector denoting the quantities of goods a firm possesses at the start of the $(t+1)$ st year.

α

β

δ A parameter in a log-linear intertemporal utility function, for the stationary-state time path, related to the subjective rate of time discounting.

$\delta(t)$ A parameter in a log-linear intertemporal utility function related to the subjective rate of time discounting.

γ A parameter in a log-linear intertemporal utility function, for the stationary-state time path, related to the agent's relative preferences for leisure and the consumer good for each year.

$\gamma(t)$ A parameter in a log-linear intertemporal utility function related to the agent's relative preferences for leisure and the consumer good for each year.

$\omega_1(t)$ The tons steel in the firm's inventory at the start of the $(t+1)$ st year.

$\omega_2(t)$ The bushels corn in the firm's inventory at the start of the $(t+1)$ st year.

$\mathbf{A}_\eta$, $\eta = \alpha, \beta$. Leontief input-output matrix for the η technique. The ith, jth element is the quantity of the ith commodity required as input for unit output in the jth industry.

$C(t)$ The deviation of bushels corn consumed each year from the stationary state consumption.

$I(t)$ The value, in bushels corn, of gross investment at the end of the *t*th year.

$L(t)$ The deviation of labor employed each year from the stationary state employment.

$L_D(t)$ The person-years of labor firms want to employ during the *t*th year.

$L_S(t)$ The person-years of labor supplied by agents during the *t*th year.

L_s The stationary-state yearly labor supply.

$P_1(t)$ The deviation of the price of steel from the stationary state price of steel.

$\mathbf{Q}(t)$ The deviation of gross outputs of steel and corn from stationary state gross outputs.

$R(t)$ The deviation of the rate of profits from the stationary state rate of profits.

S Stationary state savings.

$S(t)$ The value, in bushels corn, of gross savings at the end of the *t*th year.

$U\left(c_0, 1-l_0, c_1, 1-l_1, \ldots, c_{n-1}, 1-l_{n-1}\right)$

$U\left[c_0(t), 1-l_0(t), c_1(t), 1-l_1(t); \delta(t), \gamma(t)\right]$ The utility function of the agent born at the start of the *t*th year.

$W(t)$ The deviation of the wage from the stationary state wage.

$X_1^\eta(t)$, $\eta = \alpha, \beta$. The tons of steel produced by a firm with the η process during the *t*th year. The output of steel is available at the end of the year.

$X_2(t)$ The bushels corn produced by a firm during the *t*th year. The output of corn is available at the end of the year.

$X_3(t)$ Value of the firm's unspent endowment in the *t*th year.

$\mathbf{a}_{\bullet j}^{\eta}$, j = 1, 2; $\eta = \alpha, \beta$. Two-element column vector comprising the jth column of $\mathbf{A}_{\eta}$.

$\mathbf{a}_0^{\eta}$ $\eta = \alpha, \beta$. Two-element row vector of labor coefficients for the η technique. The jth element is the person-years of labor input for the unit output in the jth industry.

c^* Bushels corn consumed at the end of each year in a stationary state.

$c_0(t)$ Bushels corn consumed at the end of the *t*th year by the agent born at the start of the *t*th year.

$c_1(t)$ Bushels corn consumed at the end of the (t + 1)st year by the agent born at the start of the *t*th year.

c_i, c_i^* Stationary-state bushels corn consumed by an agent at the end of the *i*th year after his birth.

$c(t)$ Bushels corn consumed at the end of the *t*th year.

$\mathbf{e}_j$ The jth column of the identity matrix.

$i^*(1+r^*;\gamma,\delta)$ Stationary-state gross investment.

$k_0(t)$ The numeraire-value of the savings at the end of the *t*th year by the agent born at the start of the *t*th year.

k_i, k_i^* The stationary-state numeriare-value of the agent's savings at the end of the *i*th year after his birth.

l^* Person-years of labor employed each year in a stationary state.

$l(t)$ Person-years of labor employed each year.

$l_0(t)$ Person-years of labor supplied during the *t*th year by the agent born at the start of the *t*th year.

$l_1(t)$ Person-years of labor supplied during the (t + 1)st year by the agent born at the start of the *t*th year.

l_i, l_i^* Stationary-state person-years of labor supplied by the agent during the *i*th year after his birth.

$\mathbf{p}^*$ A two-element row vector denoting the spot prices of steel and corn quoted on the market at the end of each year in a stationary state. Since corn is the numeraire, the second element is unity.

$\mathbf{p}(t)$ A two-element row vector denoting the spot prices of steel and corn quoted on the market at the end of the *t*th year. Since corn is the numeraire, the second element is unity.

$\mathbf{q}^*$ A two element column vector denoting the quantities of steel and corn produced and available at the end of each year in a stationary state.

$\mathbf{q}(t)$ A two element column vector denoting the quantities of steel and corn produced and available at the end of the *t*th year.

r, r^* The interest rate in a stationary state on a bushel corn borrowed at the start of the year and paid at the end of the year. Also known as the rate of profits.

$r(t)$ The interest rate on a bushel corn borrowed at the start of the *t*th year and paid at the end of the *t*th year. Also known as the rate of profits.

$s^*(1+r^*;\gamma,\delta)$ Stationary state savings.

w, $w*$ The wage, in bushels corn, paid at the end of each year in a stationary state for a person-year employed during the year.

$w(t)$ The wage, in bushels corn, paid at the end of the *t*th year for a person-year employed during the *t*th year.

2.0 An Overlapping Generations Model

Assume net output, in the numeric example of a very simple temporary equilibrium model, consists of steel, measured in tons, and corn, measured in bushels. Corn is the only consumption good in this economy. Both commodities, steel and corn, can be used as capital goods in producing more steel or corn. Table 2-1 defines the technology[19] available to the firms in this economy. The managers of the firms know of only one process for producing corn, which requires inputs of labor, steel, and corn. Steel can be produced by either one of two processes, and those processes also require inputs of labor, steel, and corn. All production processes require a year to complete, exhibit Constant Returns to Scale (CRS), and totally use up their inputs.[20] Outputs of each process become available at the end of the year.

A technique is formed by a combination of a steel-producing process and the corn-producing process. Two techniques, alpha and beta, can be constructed from the given technology. Each technique is named for the selected steel-producing process in the technique. Each technique is specified by a row vector representing the labor inputs in the two processes comprising the technique:

$$\mathbf{a}_0^{\alpha} = \begin{bmatrix} a_{01}^{\alpha} & a_{02}^{\alpha} \end{bmatrix} = \begin{bmatrix} 1 & 1 \end{bmatrix} \tag{2-1}$$

$$\mathbf{a}_0^{\beta} = \begin{bmatrix} a_{01}^{\beta} & a_{02}^{\beta} \end{bmatrix} = \begin{bmatrix} \frac{6,303}{1,885} & 1 \end{bmatrix} \tag{2-2}$$

and a 2x2 Leontief matrix containing the remaining coefficients of production:

$$\mathbf{A}_{\alpha} = \begin{bmatrix} \mathbf{a}_{\bullet 1}^{\alpha} & \mathbf{a}_{\bullet 2}^{\alpha} \end{bmatrix} = \begin{bmatrix} a_{11}^{\alpha} & a_{12}^{\alpha} \\ a_{21}^{\alpha} & a_{22}^{\alpha} \end{bmatrix} = \begin{bmatrix} \frac{137}{13,860} & \frac{377}{13,860} \\ \frac{93,347}{37,700} & \frac{11}{100} \end{bmatrix} \tag{2-3}$$

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[19] The coefficients of production are defined, as described in Vienneau (2005b), to yield "nice" fractions for stationary-state switch points and the intercepts of the wage-profits curves with the wage axis.

[20] So this is a circulating capital model. Furthermore, no joint production is assumed.

Table 2-1: Available Production Processes

	Steel Processes		Corn Process
Inputs	α	β	
Labor	1 Person-Year	$3\frac{648}{1,885}$ Person-Years	1 Person-Year
Steel	$\frac{137}{13,860}$ Ton	$\frac{2,173}{6,300}$ Ton	$\frac{377}{13,860}$ Ton
Corn	$2\frac{17,947}{37,700}$ Bushels	$\frac{7,733}{188,500}$ Bushel	$\frac{11}{100}$ Bushel
Output	1 Ton Steel	1 Ton Steel	1 Bushel Corn

$$\mathbf{A}_\beta = \begin{bmatrix} \mathbf{a}_{\bullet 1}^\beta & \mathbf{a}_{\bullet 2}^\beta \end{bmatrix} = \begin{bmatrix} a_{11}^\beta & a_{12}^\beta \\ a_{21}^\beta & a_{22}^\beta \end{bmatrix} = \begin{bmatrix} \frac{2,173}{6,300} & \frac{377}{13,860} \\ \frac{7,733}{188,500} & \frac{11}{100} \end{bmatrix} \tag{2-4}$$

The model is easily generalized to a choice among any finite number of techniques. The choice of several processes can be used to construct a discrete approximation to a continuously differentiable "smooth" production function.

2.1 Production and Prices

For what prices will competitive profit-maximizing firms adopt processes consistent with the economy being in a state in which all needed capital goods continue to be (re)produced? An analysis of the firm based on linear programming can answer this question.[21] Accordingly, consider a firm that starts the year with an inventory consisting of $\omega_1(t-1)$ tons steel and $\omega_2(t-1)$ bushels corn. Let $\Omega^T(t-1) = \begin{bmatrix} \omega_1(t-1) & \omega_2(t-1) \end{bmatrix}$. The firm faces prices of steel and corn of $\mathbf{p}(t-1) = \begin{bmatrix} p_1(t-1) & 1 \end{bmatrix}$ at the beginning of the year *t* and prices of steel and corn of $\mathbf{p}(t) = \begin{bmatrix} p_1(t) & 1 \end{bmatrix}$ at the end of the year.[22] Assume that laborers hired to work during the year are paid the wage $w(t)$ at the end of the year. The managers of the firm set the levels of operation, $X_1^\alpha(t)$ and $X_1^\beta(t)$, respectively, of the two steel-producing processes and the level of operation, $X_2(t)$, of the corn-producing process to maximize the increment of the value of the firm. They are subject to the constraint that they can purchase the required inputs out of the value of the endowment at the start of the year. Display 2-5 sets out a Linear Program (LP) specifying the competitive profit-maximizing firm's problem:

[21] The derivation in this section generalizes the approach in Vienneau (2005a).
[22] Corn is the numeraire throughout this paper.

Table 2-2: Solution of Primal LP

Variable in Basis	Value of Basic Variable	When Optimal
$X_1^{\alpha}(t)$	$\frac{\mathbf{p}(t-1)\cdot\Omega(t-1)}{\mathbf{p}(t-1)\cdot\mathbf{a}_{\bullet 1}^{\alpha}}$	$\frac{p_1(t)-a_{01}^{\beta}w(t)}{\mathbf{p}(t-1)\cdot\mathbf{a}_{\bullet 1}^{\beta}} \leq \frac{p_1(t)-a_{01}^{\alpha}w(t)}{\mathbf{p}(t-1)\cdot\mathbf{a}_{\bullet 1}^{\alpha}}$ $\frac{1-a_{02}^{\alpha}w(t)}{\mathbf{p}(t-1)\cdot\mathbf{a}_{\bullet 2}^{\alpha}} \leq \frac{p_1(t)-a_{01}^{\alpha}w(t)}{\mathbf{p}(t-1)\cdot\mathbf{a}_{\bullet 1}^{\alpha}}$ $\mathbf{p}(t-1)\cdot\mathbf{a}_{\bullet 1}^{\alpha}+a_{01}^{\alpha}w(t)\leq p_1(t)$
$X_1^{\beta}(t)$	$\frac{\mathbf{p}(t-1)\cdot\Omega(t-1)}{\mathbf{p}(t-1)\cdot\mathbf{a}_{\bullet 1}^{\beta}}$	$\frac{p_1(t)-a_{01}^{\alpha}w(t)}{\mathbf{p}(t-1)\cdot\mathbf{a}_{\bullet 1}^{\alpha}} \leq \frac{p_1(t)-a_{01}^{\beta}w(t)}{\mathbf{p}(t-1)\cdot\mathbf{a}_{\bullet 1}^{\beta}}$ $\frac{1-a_{02}^{\alpha}w(t)}{\mathbf{p}(t-1)\cdot\mathbf{a}_{\bullet 2}^{\alpha}} \leq \frac{p_1(t)-a_{01}^{\beta}w(t)}{\mathbf{p}(t-1)\cdot\mathbf{a}_{\bullet 1}^{\beta}}$ $\mathbf{p}(t-1)\cdot\mathbf{a}_{\bullet 1}^{\beta}+a_{01}^{\beta}w(t)\leq p_1(t)$
$X_2(t)$	$\frac{\mathbf{p}(t-1)\cdot\Omega(t-1)}{\mathbf{p}(t-1)\cdot\mathbf{a}_{\bullet 2}^{\alpha}}$	$\frac{p_1(t)-a_{01}^{\eta}w(t)}{\mathbf{p}(t-1)\cdot\mathbf{a}_{\bullet 1}^{\eta}} \leq \frac{1-a_{02}^{\alpha}w(t)}{\mathbf{p}(t-1)\cdot\mathbf{a}_{\bullet 2}^{\alpha}},\ \eta=\alpha,\beta$ $\mathbf{p}(t-1)\cdot\mathbf{a}_{\bullet 2}^{\alpha}+a_{02}^{\alpha}w(t)\leq 1$
$X_3(t)$	$\mathbf{p}(t-1)\cdot\Omega(t-1)$	$\mathbf{p}(t-1)\cdot\mathbf{a}_{\bullet 1}^{\eta}+a_{01}^{\eta}w(t)\geq p_1(t),\ \eta=\alpha,\beta$ $\mathbf{p}(t-1)\cdot\mathbf{a}_{\bullet 2}^{\alpha}+a_{02}^{\alpha}w(t)\geq 1$

Choose $X_1^{\alpha}(t)$, $X_1^{\beta}(t)$, $X_2(t)$

To Maximize $\sum_{\eta=\alpha,\beta}\left[p_1(t)-\mathbf{p}(t-1)\cdot\mathbf{a}_{\bullet 1}^{\eta}-a_{01}^{\eta}w(t)\right]X_1^{\eta}(t)$

$$+\left[1-\mathbf{p}(t-1)\cdot\mathbf{a}_{\bullet 2}^{\alpha}-a_{02}^{\alpha}w(t)\right]X_2(t)$$

Such That (2-5)

$$\sum_{\eta=\alpha,\beta}\mathbf{p}(t-1)\cdot\mathbf{a}_{\bullet 1}^{\eta}\,X_1^{\eta}(t)+\mathbf{p}(t-1)\cdot\mathbf{a}_{\bullet 2}^{\alpha}\,X_2(t)\leq\mathbf{p}(t-1)\cdot\Omega(t-1)$$

$$X_1^{\alpha}(t)\geq 0,\ X_1^{\beta}(t)\geq 0,\ X_2(t)\geq 0$$

Display 2-6 specifies the dual LP:

Choose $r(t)$

To Minimize $r(t)\mathbf{p}(t-1)\cdot\Omega(t-1)$

Such That (2-6)

$$\mathbf{p}(t-1)\cdot\mathbf{a}_{\bullet 1}^{\eta}\left[1+r(t)\right]+a_{01}^{\eta}w(t)\geq p_1(t),\ \eta=\alpha,\ \beta$$

$$\mathbf{p}(t-1)\cdot\mathbf{a}_{\bullet 2}^{\alpha}\left[1+r(t)\right]+a_{02}^{\alpha}w(t)\geq 1$$

$$r(t) \geq 0$$

Competition ensures that the corn own-rate of interest minimizes the cost of production. A constraint in the dual LP is met with equality if the corresponding decision variable is positive in the primal LP.

The solution to the primal LP (Table 2-2) determines which production processes, if any, firms are willing to operate, given any configuration of wages and the price of steel. Consider dynamic paths in which firms find it profitable in each period to continue to produce the capital goods (steel and corn) that are needed to allow the economy to continue. Given the current-year wage and the spot price of steel at the start of the year, one can calculate the spot price of steel along such a path. Likewise, the dual LP shows how to calculate the rate of profits. One still needs to find the initial spot price of steel; the wage; and how much steel and corn is produced each year, given the initial quantities of steel and corn. Only certain ranges of initial values and each yearly wage are compatible with dynamic paths in which the economy continues to be reproduced.

2.2 Quantity Flows

Section 2.1 shows how to find the optimal technique along a dynamic path in which firms produce steel and corn each year. Let $\mathbf{q}(t)$ be a column vector denoting the levels of operation in year *t* of the two processes comprising the technique. In other words, $\mathbf{q}(t)$ gross outputs are available at the end of the *t*th year, aggregating over firms, from the selected technique. The capital goods, $[\mathbf{q}(t) - c(t)\mathbf{e}_2]$, available at the start of the next year are the difference between these gross outputs and the corn consumed. $c(t)$ is the bushels corn consumed at the end of the *t*th year, and $\mathbf{e}_2$ is the second column of the identity matrix. The value of gross investment, in numeraire units, is merely the value of these capital goods:

$$I(t) = \mathbf{p}(t) \cdot [\mathbf{q}(t) - c(t)\mathbf{e}_2] \quad (2\text{-}7)$$

If one considers only paths in which neither steel nor corn is in excess supply, gross outputs evolve as shown in Equation 2-8:[23]

$$\mathbf{q}(t-1) - c(t-1)\mathbf{e}_2 = \mathbf{A}_\eta \cdot \mathbf{q}(t),\ \eta = \alpha, \beta \quad (2\text{-}8)$$

Equation 2-9 gives the quantity of labor that must be hired to operate each technique at these levels:

[23] When prices are such that profit-maximizing firms are willing to operate either of the two steel-producing processes, as well as the corn-producing process, quantities evolve as a linear combination of Equation 2-8, as defined for each technique. Likewise, linear combinations should be considered for Equation 2-9.

$$L_D(t) = \mathbf{a}_0^{\eta} \cdot \mathbf{q}(t)\,,\;\; \eta = \alpha, \beta, \qquad (2\text{-}9)$$

where $L_D(t)$ is the person-years of labor demanded during the *t*th year.

Initial quantities of steel and corn are part of the data for this model. Equation 2-8 only determines the outputs of steel and corn after the start of the initial year, given the technique(s) and the corn consumed each year. Consumption choices, as well as the amount of labor supplied each year, have not yet been specified.

2.3 Consumption and Leisure

Introducing utility-maximization within an overlapping generations model is one way to close the model and define a full equilibrium of the economy.[24] Assume each worker in the overlapping generations model lives two years. Each worker is born at the beginning of a year, sells a person-year of labor services for use during the first year of his life, purchases some corn to consume immediately out of the wages paid at the end of the year, and saves the remainder of his wages to be used to purchase corn for consumption at the end of the last year of his life (Figure 1). Likewise, the agent can work during the second year of his life for wages paid at the end of that year.

Even further, assume a single worker is born each year.[25] Consumption, $c(t)$, out of the output of a given year is the sum of the consumption of the agent born at the start of that year and the agent born at the start of the previous year:

$$c(t) = c_0(t) + c_1(t-1), \qquad (2\text{-}10)$$

where $c_0(t)$ is the consumption of the agent working in year *t*, and $c_1(t-1)$ is the consumption of the agent working in year *t* – 1. The labor supplied, $L_S(t)$, each year is similarly the sum of the labor supplied by the agents alive during that year:

$$L_S(t) = l_0(t) + l_1(t-1), \qquad (2\text{-}11)$$

[24] The closure adopted here follows the neoclassical models in Marglin (1984) and in Ferreti (2004) and Bloise and Reichlin (2005). Another utility-maximizing closure uses agents with infinite lives (Schefold 2000). Marglin also describes a Marxian closure in which the wage is exogeneous. Certain monetary theories of distribution (e.g., Panico 1999) take the rate of profits as determined by the policy of the monetary authority. The Cambridge equation (Kaldor 1955-56 and 1966, Kahn 1959, Pasinetti 1962, and Robinson 1962) provides a long-period Post Keynesian closure.

[25] Kirman (1992) shows that assumption of a representative individual is unjustified in general. Here the assumption abstracts from the difficulties addressed by Kirman; yet the potential for interesting dynamics is shown to arise anyways.

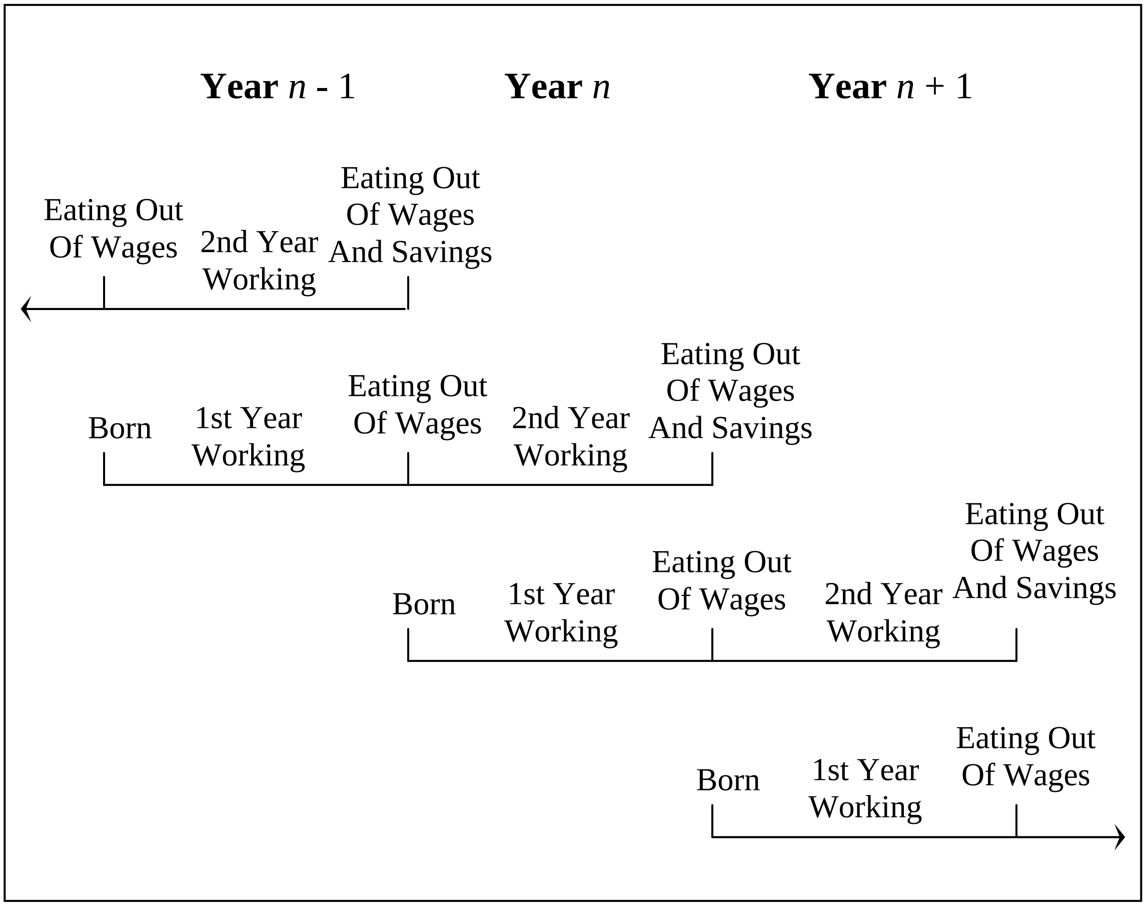

Figure 2-1: Overlapping Generations

where $l_0(t)$ is the labor supplied during the *t*th year by the agent born at the start of that year, and $l_1(t-1)$ is the labor supplied during the *t*th year by the agent born the previous year.

For simplicity and concreteness, assume each agent has a log-linear utility function. Wages earned by a worker during the first year of his life and not consumed immediately are saved at the going interest rate. They are consumed at the end of the second year of his life. Thus, each worker faces two budget constraints, one for each year of his life. One constraint is that his consumption and savings at the end of the first year of his life add up to the wages earned during that year. The second constraint is that his consumption at the second year of his life be the sum of wages earned during that year and the value of his savings, where the accumulated due to interest is included in the value of his savings. Each agent, facing a given wage and interest rate, chooses his lifecycle consumption and labor supply to solve the mathematical programming problem in Display 2-12:

Choose $c_0(t)$, $1-l_0(t)$, $k_0(t)$, $c_1(t)$, $1-l_1(t)$

To maximize $U\left[c_0(t), 1-l_0(t), c_1(t), 1-l_1(t); \delta(t), \gamma(t)\right]$

$$= \ln\left[c_0(t)\right] + \gamma(t)\ln\left[1-l_0(t)\right] + \frac{\ln\left[c_1(t)\right] + \gamma(t)\ln\left[1-l_1(t)\right]}{1+\delta(t)} \tag{2-12}$$

Such that $c_0(t) + k_0(t) = w(t)l_0(t)$

$$c_1(t) = w(t+1)l_1(t) + [1 + r(t+1)]k_0(t)$$

$$c_0(t) \geq 0,\ l_0(t) \geq 0,\ k_0(t) \geq 0,\ c_1(t) \geq 0,\ l_1(t) \geq 0$$

This programming problem is formulated in terms of consumption and leisure choices, instead of an equivalent formulation in terms of consumption and labor supply choices. That is, it is formulated in terms of desired goods, instead of a mixture of goods and "bads".

The specification of the budget constraints embodies a couple of modeling assumptions. First, the use of two constraints, instead of only one, reflects the assumption that households cannot borrow from future consumption. No market exists for bonds in which households can sell, at the end of their first year, a promise to provide a specified value of corn at the end of the second year.[26] Households can, however, defer consumption from the end of their first year to the second year by loaning capital to firms. Note that only the value of the capital stocks, $k_0(t)$, enters the household's utility-maximization problem. Holdings of steel and corn are perfect substitutes in equilibrium for savings. This is an aspect of the law of one price.[27]

The solution of the household's utility maximization problem specifies savings and labor supply functions. A number of marginal conditions exist, but only three are independent. The marginal rate of substitution between consumption at the end of the two periods must equal, in an interior equilibrium, the slope of the relevant two-dimensional projection of the budget constraints:

$$\frac{\frac{\partial U}{\partial c_0(t)}}{\frac{\partial U}{\partial c_1(t)}} = [1+\delta(t)]\frac{c_1(t)}{c_0(t)} = 1 + r(t+1) \tag{2-13}$$

In each period, a marginal condition relates the transformation between consumption and leisure in that period:

---

[26] Relaxing this restriction does not seem to imply any dramatic change in the qualitative aspects of the model highlighted in this paper.

[27] Parrinello (2005).

$$\frac{\frac{\partial U}{\partial[1-l_0(t)]}}{\frac{\partial U}{\partial c_0(t)}}=\gamma(t)\frac{c_0(t)}{1-l_0(t)}=w(t) \tag{2-14}$$

$$\frac{\frac{\partial U}{\partial[1-l_1(t)]}}{\frac{\partial U}{\partial c_1(t)}}=\gamma(t)\frac{c_1(t)}{1-l_1(t)}=w(t+1) \tag{2-15}$$

Displays 2-13, 2-14, and 2-15 give three linear equations in four variables ($c_0(t)$, $l_0(t)$, $c_1(t)$, and $l_1(t)$). A fourth linear equation can be derived from the budget constraints:

$$[1+r(t+1)]c_0(t)+c_1(t)=[1+r(t+1)]w(t)l_0(t)+w(t+1)l_1(t) \tag{2-16}$$

Displays 2-17 through 2-20 show the solution household's utility-maximization problem.

$$c_0(t)=\left[\frac{1}{1+\gamma(t)}\right]\left[\frac{1+\delta(t)}{2+\delta(t)}\right]\left\{w(t)+\frac{w(t+1)}{[1+r(t+1)]}\right\} \tag{2-17}$$

$$c_1(t)=\left[\frac{1}{1+\gamma(t)}\right]\left[\frac{1}{2+\delta(t)}\right][1+r(t+1)]\left\{w(t)+\frac{w(t+1)}{[1+r(t+1)]}\right\} \tag{2-18}$$

$$l_0(t)=1-\left[\frac{\gamma(t)}{1+\gamma(t)}\right]\left[\frac{1+\delta(t)}{2+\delta(t)}\right]\left\{1+\left[\frac{1}{1+r(t+1)}\right]\left[\frac{w(t+1)}{w(t)}\right]\right\} \tag{2-19}$$

$$l_1(t)=1-\left[\frac{\gamma(t)}{1+\gamma(t)}\right]\left[\frac{1}{2+\delta(t)}\right]\left\{1+[1+r(t+1)]\left[\frac{w(t)}{w(t+1)}\right]\right\} \tag{2-20}$$

Desired savings at the end of each year is easily derived. Neither the generation expiring nor the generation born at the end of the *t*th year has any savings. Hence the value of savings at the end of the *t*th year is the value of the savings out of wages of the generation born at the start of that year:

$$S(t)=k_0(t)=\frac{c_1(t)-w(t+1)l_1(t)}{1+r(t+1)} \tag{2-21}$$

Or:

$$S(t) = \left[\frac{1}{2+\delta(t)}\right]\left\{w(t) - [1+\delta(t)]\frac{w(t+1)}{[1+r(t+1)]}\right\} \tag{2-22}$$

Equation 2-11 expresses the labor supply as the sum of the labor supplies of generations alive each year. The economy-wide labor supply can be expressed as a function of the time paths of wages, the rate of profits, and the parameters of the utility functions:

$$L_S(t) = 2 - \left[\frac{\gamma(t)}{1+\gamma(t)}\right]\left[\frac{1+\delta(t)}{2+\delta(t)}\right]\left\{1+\left[\frac{1}{1+r(t+1)}\right]\left[\frac{w(t+1)}{w(t)}\right]\right\}$$
$$-\left[\frac{\gamma(t-1)}{1+\gamma(t-1)}\right]\left[\frac{1}{2+\delta(t-1)}\right]\left\{1+[1+r(t)]\left[\frac{w(t-1)}{w(t)}\right]\right\} \tag{2-23}$$

2.4 Equilibrium in the Capital and Labor Markets

At equilibrium prices, households want to hold the entire stock of capital:

$$I(t) = S(t) \tag{2-24}$$

Firms choose to hire the labor supplied by households:

$$l(t) = L_D(t) = L_S(t), \tag{2-25}$$

where $l(t)$ is the quantity of labor employed in equilibrium. The equilibrium conditions in Equations 2-24 and 2-25 close the model. Equation 2-24 is a stock equilibrium condition in which the aggregate value of capital appears. This value is endogenous; it is not taken as data. Also, the marginal product of the value of capital plays no role in closing the model. This marginal product is not equated to the interest rate.[28] One should not use the investment function or the labor demand function in Equations 2-24 and 2-25 for the analysis of tatonnement stability without further analysis. In particular, the factor demand functions in such analysis should be derived demand functions, incorporating out-of-equilibrium consumption decisions by households.[29]

2.5 Initial Conditions

Consider an equilibrium path starting at the end of year zero or, equivalently, at the start of the first year. Neither the firms nor the households can alter past results. Bygones are bygones. What results of past decisions and actions must be taken as given in specifying the model?

---

[28] Cost minimization, as described in Section 2.1, ensures that the rental price of each capital good is equal to the value of the marginal product of that capital good in each industry, where marginal products are intervals defined by the right-hand and left-hand derivatives of the production function. See Hahn (1982).
[29] Ferretti (2004) and Marglin (1984).

Presumably firms were engaged in production during year zero. At any rate, $\mathbf{q}(0)$, the tons of steel and bushels of corn available at the end of year zero, are part of the given initial conditions.

Three generations of households exist at the end of year zero. There is a generation just expiring, the generation that has worked for a year and may decide to work for the initial year in the model, and the generation just being born. The generation just expiring affects the future path insofar as their consumption of corn must be subtracted from the initial quantity, $q_2(0)$, of corn. But all the variables that determine this consumption are in the past and are taken as given data. In particular, the following are given:

- The value of this generation's savings, $k_0(-1)$, at the beginning of year zero.
- The amount of this generation's labor, $l_1(-1)$, used during year zero.
- The wage, $w(0)$, paid at the end of year zero, for which this labor was contracted.
- The rate of profits, $r(0)$, during year zero.

This first generation's consumption at the end of year zero is accordingly determined, as specified in Equation 2-26:

$$c_1(-1) = w(0)\, l_1(-1) + [1 + r(0)] k_0(-1) \qquad \text{(2-26)}$$

Consider the generation born at the start of year zero. The amount of labor, $l_0(0)$, supplied by this generation during year zero and the wage, $w(0)$, are part of the given initial conditions. Consequently, the wages, $w(0)\, l_0(0)$, paid to this generation at the end of year zero are determined by the given data. The household of this generation must still decide how to apportion his current year's wages between consumption and savings, and how much leisure or consumption he wants for the first year of the model. Accordingly, this generated is modeled as solving the mathematical programming problem in Display 2-27, in which the labor supplied in year zero does not appear as a decision variable:

Choose $c_0(0)$, $k_0(0)$, $c_1(0)$, $1 - l_1(0)$

To maximize $U\left[c_0(0), c_1(0), 1 - l_1(0); \delta(0), \gamma(0)\right]$

$$= \ln\left[c_0(0)\right] + \frac{\ln\left[c_1(0)\right] + \gamma(0) \ln\left[1 - l_1(0)\right]}{1 + \delta(0)} \qquad \text{(2-27)}$$

Such that $c_0(0) + k_0(0) = w(0)\, l_0(0)$

$c_1(0) = w(1)\, l_1(0) + [1 + r(1)] k_0(0)$

$c_0(0) \geq 0,\ k_0(0) \geq 0,\ c_1(0) \geq 0,\ l_1(0) \geq 0$

The point is the labor-leisure trade off cannot be made in year zero. The right-hand-side of the first constraint is given.

The generation just being born at the start of the first year is completely modeled by the analysis in Section 2.3. All generations completely in the model, including the generation born at the start of the first year, have correct expectations. The generation dying at the end of the first year has correct expectations from the end of year zero forward. No assumption is made that this generation's expectations were correct at the start of year zero. Likewise, no condition is imposed on expectations prior to the end of year zero for any other agents.

The assumption that firms find it profitable to produce both steel and corn along equilibrium paths found by solving the model constrains the range of initial values.

2.6 Recap

This section has presented a model of dynamic equilibrium paths. In this model, the givens are:

- Technology, as specified by the coefficients of each steel-producing and corn-producing process.
- The tastes of the representative agent for each generation, as embodied in the parameters, $\gamma(t)$ and $\delta(t)$, of utility functions.
- The values of the agents' decision variables in years prior to first year in the dynamic equilibrium path found by solving the model. In particular, the data[30] include:
    - Initial quantities, $\mathbf{q}(0)$, of steel and corn.
    - The labor supplied by agents, $l_1(-1)$ and $l_0(0)$, during the year prior to the first.
    - The numeraire-value, $k_0(-1)$ and 0, of savings at the start of that year for households that were alive before the first year of the model and are still alive at the start of the first year in the model.
- Price variables prior to those in the solution dynamic equilibrium path. In particular, the data include:
    - The wage, $w(0)$, for which households agree to work during the year prior to the first year in the solution dynamic equilibrium path.
    - The rate of profits, $r(0)$, during that year.

In summary, tastes, technology, and history are taken as given data. History yields the initial endowments and constrains the choices available to agents in the model. A solution to the model is a dynamic equilibrium path. Along such a path, the following are specified:

[30] The entire prior history of decision and price variables before the start of the model could be specified as exogenous variables. Only those listed, however, are relevant for the solution of the model.

- The technique(s) chosen each year.
- The quantities of steel and corn produced each year ($\mathbf{q}(t)$, t = 1, 2, …), including during the initial year of the dynamic equilibrium path found by solving the model.
- The quantities of labor employed each year ($l(t)$, t = 1, 2, …), including the decomposition of labor among the household generations alive each year.
- The amount of corn consumed each year ($c(t)$, t = 0, 1, 2, …), including its decomposition each year among the generations.
- The dynamic path of the price ($p_1(t)$, t = 0, 1, 2, …) of steel (including the initial price), wages ($w(t)$, t = 1, 2, …), and the rate of profit ($r(t)$, t = 1, 2, …).
- The allowable ranges for initial values thrown up by history such that a dynamic path exists in which steel, corn, and labor are never in excess supply and firms choose to produce steel and corn each year.

Expected prices, as separate variables, do not enter this model. In effect, firms and households are assumed to correctly anticipate all variables along a dynamic equilibrium path.

Two types of stability analysis are important in analyzing this model. First, the nature of the long run behavior of the solution dynamic equilibrium paths is of interest. Do limit points, that is, stationary states exist? Do at least some of the dynamic equilibrium paths converge to these limit points? What happens to paths that diverge? Are there limit cycles or some other sort of attractors? Second, the stability of dynamic equilibrium paths themselves can be analyzed in some sort of extension of this model. One extension would be to assume some sort of tatonnement process at the start of each year.[31] What characteristics of solutions of the original model have implications for tatonnement stability? The separation between the process of achieving equilibrium and movement along equilibrium paths is one aspect of the model that sets it in logical time, instead of historical time.[32]

This model permits the analysis of some thought experiments, throwing light on some traditional neoclassical beliefs. Consider a case in which $\gamma(t)$ is a constant function, but $\delta(t)$ is a non-increasing, non-constant function. In this case, the consumers increase the supply of capital, ceteris paribus. If stationary-state dynamic equilibrium paths are stable in this model, Cambridge capital-reversing examples correspond to paths, in this case, in which the cost-minimizing technique varies along the path and both the value of capital per worker and the interest rate increase together. On the other hand, consider a case in which $\gamma(t)$ is a non-increasing, non-constant function, and $\delta(t)$ is a constant. Then the consumers increase the supply of labor, ceteris paribus. Capital-reversing results here in paths in which both the labor intensity of the cost-minimizing technique and the wage increase together.

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[31] This is the approach of Hicks (1946). Hicks calls each production period a "week", while I, following Sraffa (1960), call them a "year".
[32] Robinson (1974).

3.0 Stationary States

Now that a short run model has been specified, the dynamic equilibrium paths determined by the model can begin to be analyzed. The goal of the analysis is to find evidence for paths illustrating behavior counter to intuition trained by traditional neoclassical theory and to explore whether capital-theoretical "paradoxes" in Cambridge models can help in revealing such counterintuitive behavior. Mathematicians often begin their analysis of dynamical systems by finding limit points (Guckenheimer and Holmes 1983).

Such points are easy to find in this model if all agents have identical tastes, no matter in which generation they are born. Accordingly, assume the parameters of the utility function are constant through time:

$$\delta(t) = \delta,\ t = 0, 1, 2, \ldots \tag{3-1}$$

$$\gamma(t) = \gamma,\ t = 0, 1, 2, \ldots \tag{3-2}$$

Under this condition, limit points are stationary states. That is, one drops the time index from all state variables, for example:

$$\mathbf{p}(t) = \mathbf{p}^*,\ t = 0, 1, 2, \ldots \tag{3-3}$$

$$w(t) = w^*,\ t = 1, 2, 3, \ldots \tag{3-4}$$

$$r(t) = r^*,\ t = 1, 2, 3, \ldots \tag{3-5}$$

$$\mathbf{q}(t) = \mathbf{q}^*,\ t = 0, 1, 2, \ldots \tag{3-6}$$

$$c(t) = c^*,\ t = 0, 1, 2, \ldots \tag{3-7}$$

$$l(t) = l^*,\ t = 0, 1, 2, \ldots \tag{3-8}$$

A "star" over a variable indicates the value of the variable in a stationary state.

3.1 Prices and the Choice of Technique

The assumption of stationary prices simplifies the primal LP (Display 2-5) that, when solved, shows the production processes profit-maximizing firms are willing to adopt. Figure 3-1 shows a graph of the regions defined by the inequalities in the third column of Table 2-2. The heavy locus shown on this graph defines the stationary price of steel as a function of the wage along a steady state path in which both steel and corn continue to be reproduced. This graph also shows which steel-producing process is adopted, thereby showing the choice of technique as a function of the wage. Since the alpha technique is adopted at high and low wages, while the beta technique is adopted at middling wages, the example illustrates *reswitching*.

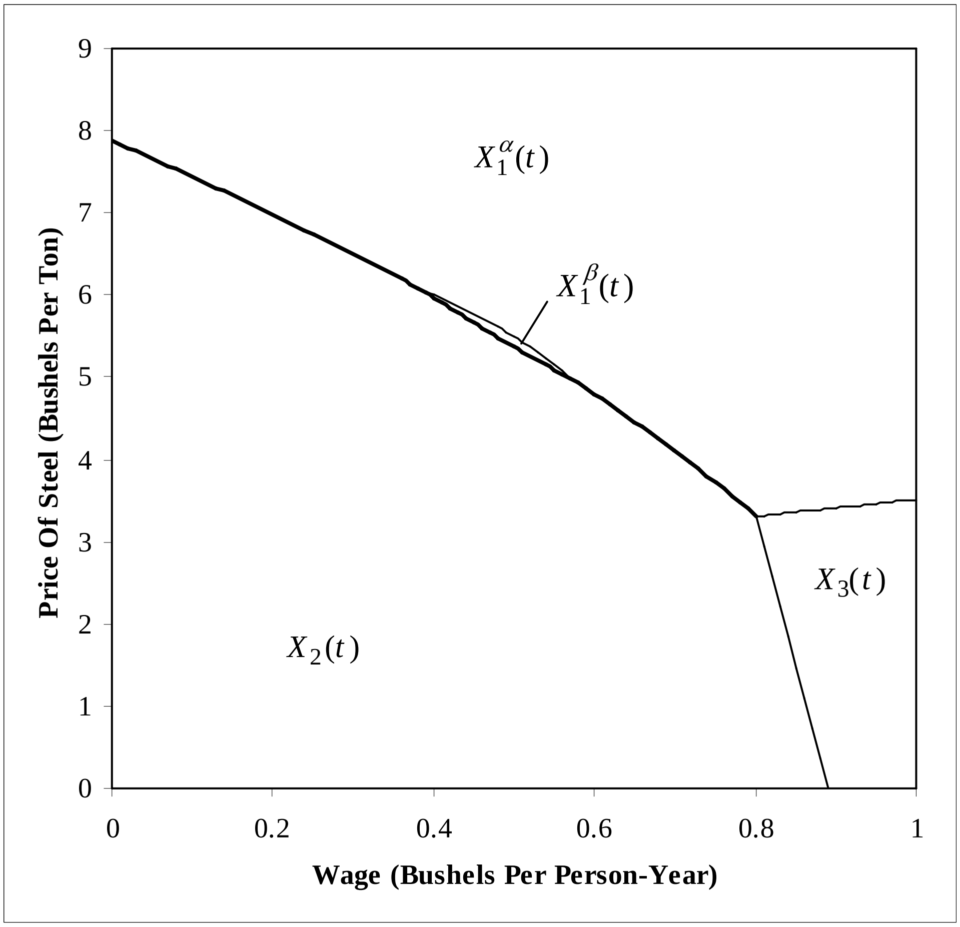

Figure 3-1: Stationary State Profit-Maximizing Processes

For each production process in a technique, the corresponding constraint in the dual LP (Display 2-6) is met with equality. Thus, each technique yields a system of two equations:

$$\mathbf{p}^* \cdot \mathbf{A}_\eta (1 + r^*) + \mathbf{a}_0^\eta w^* = \mathbf{p}^*, \ \eta = \alpha, \beta \tag{3-9}$$

Since corn is the numeraire, Equation 3-10 follows[33] from Equation 3-9:

$$1 = \mathbf{p}^* \cdot \mathbf{e}_2 = \mathbf{a}_0^\eta \cdot \left(\mathbf{I} - (1 + r^*)\mathbf{A}_\eta\right)^{-1} \cdot \mathbf{e}_2 w^*, \ \eta = \alpha, \beta \tag{3-10}$$

Or:

[33] The Hawkins-Simon condition ensures that the matrix inverse exists for a closed and connected range of the rate of profits including zero.

$$w^* = \frac{1}{\mathbf{a}_0^\eta \cdot \left(\mathbf{I} - (1+r)\mathbf{A}_\eta\right)^{-1} \cdot \mathbf{e}_2}, \ \eta = \alpha, \beta$$

$$= \frac{(a_{11}^\eta a_{22}^\eta - a_{12}^\eta a_{21}^\eta)(1+r^*)^2 - (a_{11}^\eta + a_{22}^\eta)(1+r^*) + 1}{(a_{12}^\eta a_{01}^\eta - a_{11}^\eta a_{02}^\eta)(1+r^*) + a_{02}^\eta}, \ \eta = \alpha, \beta \qquad (3\text{-}11)$$

Equation 3-11 is known as the wage-rate of profits curve for each technique. The choice of the cost-minimizing technique ensures the steady state wage and rate of profits lies on the outer envelope of all such curves. This outer envelope is known as the *wage-rate of profits frontier*. Equation 3-12 gives the wage-rate of profits frontier for the example:

$$w^* = \begin{cases} \dfrac{(14{,}100 - 4{,}373r^* - 1{,}148r^{*2})}{75(235 + 4r^*)}, & 0 \le r^* \le 75\% \quad \text{or} \quad 125\% \le r^* \le r_{\max} \\ \dfrac{1{,}833 - 1{,}201r^* + 116r^{*2}}{50(47 - 16r^*)}, & 75\% \le r^* \le 125\% \end{cases} \qquad (3\text{-}12)$$

Equation 3-13 gives the maximum rate of profits in the example:

$$r_{\max} = \frac{\sqrt{83{,}870{,}329} - 4{,}373}{2{,}296} \approx 208.4\% \qquad (3\text{-}13)$$

The wage-rate of profits curves for the techniques are plotted in Figure 3-2.

One can substitute the wage for each technique, as a function of the rate of profits, back into Equation 3-9. This allows one to determine the steady state prices of all produced commodities. Equation 3-14 gives the price of steel in the two-commodity model:

$$p_1^* = \frac{\mathbf{a}_0^\eta \cdot \left(\mathbf{I} - (1+r^*)\mathbf{A}_\eta\right)^{-1} \cdot \mathbf{e}_1}{\mathbf{a}_0^\eta \cdot \left(\mathbf{I} - (1+r^*)\mathbf{A}_\eta\right)^{-1} \cdot \mathbf{e}_2}, \ \eta = \alpha, \beta$$

$$= \frac{a_{01}^\eta - (a_{22}^\eta a_{01}^\eta - a_{21}^\eta a_{02}^\eta)(1+r^*)}{(a_{12}^\eta a_{01}^\eta - a_{11}^\eta a_{02}^\eta)(1+r^*) + a_{02}^\eta}, \ \eta = \alpha, \beta \qquad (3\text{-}14)$$

The numerical values for the example have been used in Equation 3-15:

$$p_1^* = \begin{cases} \dfrac{231(1{,}269 + 892r^*)}{377(235 + 4r^*)}, & 0 \le r^* \le 75\% \quad \text{or} \quad 125\% \le r^* \le r_{\max} \\ \dfrac{693(517 - 56r^*)}{1{,}885(47 - 16r^*)}, & 75\% \le r^* \le 125\% \end{cases} \qquad (3\text{-}15)$$

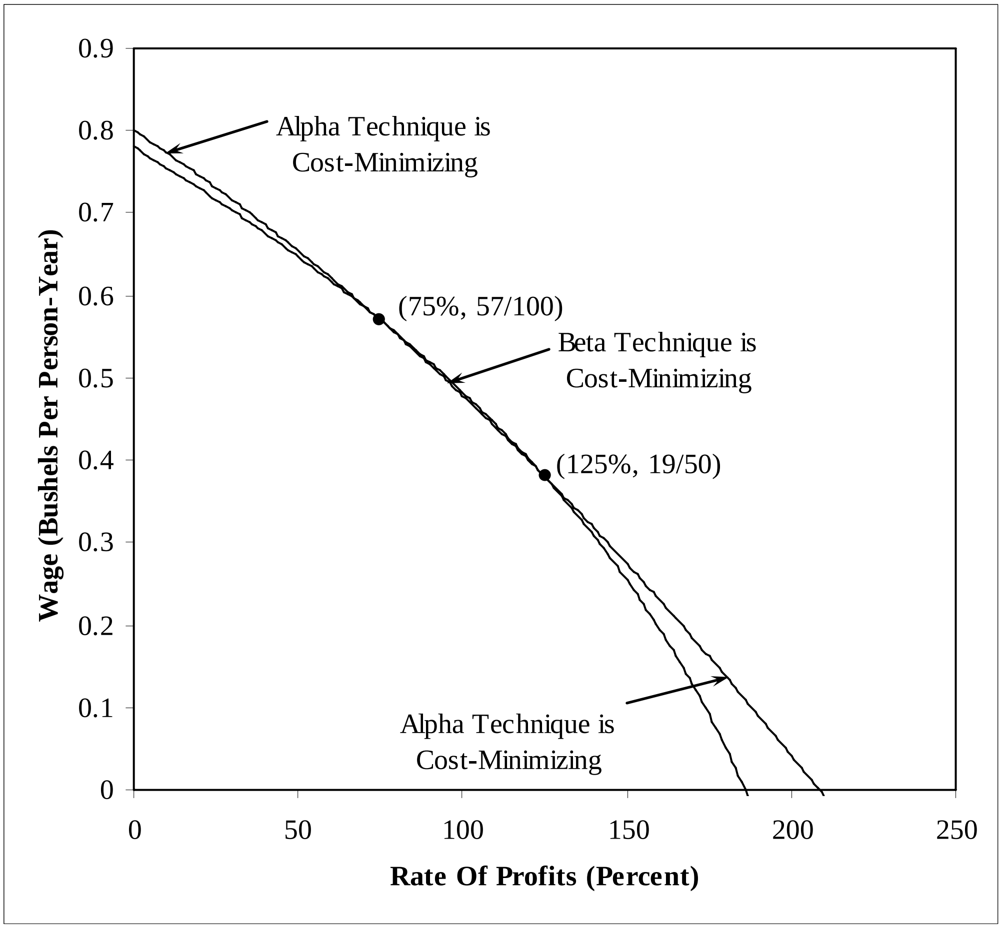

Figure 3-2: Wage-Rate of Profits Frontier

3.2 Stationary State Quantity Flows

Equation 3-16 shows Equation 2-8 specialized to stationary states:

$$\mathbf{q}^* - c^* \mathbf{e}_2 = \mathbf{A}_\eta \cdot \mathbf{q}^*,\ \eta = \alpha, \beta \qquad (3\text{-}16)$$

Some simple manipulations give Equation 3-17:

$$\mathbf{q}^* = c^* (\mathbf{I} - \mathbf{A}_\eta)^{-1} \cdot \mathbf{e}_2,\ \eta = \alpha, \beta \qquad (3\text{-}17)$$

Equation 3-18 imposes the condition that the stationary-state level of employment be adopted:

$$l^* = \mathbf{a}_0^\eta \cdot \mathbf{q}^* = \mathbf{a}_0^\eta \cdot (\mathbf{I} - \mathbf{A}_\eta)^{-1} \cdot c^* \mathbf{e}_2,\ \eta = \alpha, \beta \qquad (3\text{-}18)$$

Obvious algebra specifies consumption, in terms of the chosen technique, per person-year:

$$\frac{c^*}{l^*} = \frac{1}{\mathbf{a}_0^{\eta} \cdot (\mathbf{I} - \mathbf{A}_{\eta})^{-1} \cdot \mathbf{e}_2} = \frac{\left[(1 - a_{11}^{\eta})(1 - a_{22}^{\eta}) - a_{12}^{\eta} a_{21}^{\eta}\right]}{a_{12}^{\eta} a_{01}^{\eta} + (1 - a_{11}^{\eta}) a_{02}^{\eta}}, \ \eta = \alpha, \beta \qquad (3\text{-}19)$$

Note that for a level of employment of one person-year, Equation 3-19 is of the same form as the wage-rate of profits frontier evaluated at a rate of profits of zero.[34] Substituting Equation 3-19 for consumption into Equation 3-17 yields Equation 3-20:

$$\frac{1}{l^*}\mathbf{q}^* = \frac{(\mathbf{I} - \mathbf{A}_{\eta})^{-1} \cdot \mathbf{e}_2}{\mathbf{a}_0^{\eta} \cdot (\mathbf{I} - \mathbf{A}_{\eta})^{-1} \cdot \mathbf{e}_2} = \frac{1}{a_{12}^{\eta} a_{01}^{\eta} + (1 - a_{11}^{\eta}) a_{02}^{\eta}} \begin{bmatrix} a_{12}^{\eta} \\ 1 - a_{11}^{\eta} \end{bmatrix}, \ \eta = \alpha, \beta \qquad (3\text{-}20)$$

Equation 3-20 shows the stationary state quantity flows per person-year for each industry, where these flows generate a net output consisting solely of the specified consumption basket. Tables 3-1 and 3-2 are generated with this equation.

Table 3-1: Quantity Flows Per Person-Year for the Alpha Technique

Inputs	**Steel Industry**	**Corn Industry**
Labor	$\frac{377}{14{,}100}$ Person-Year	$\frac{13{,}723}{14{,}100}$ Person-Year
Steel	$\frac{51{,}649}{195{,}426{,}000}$ Ton	$\frac{5{,}173{,}571}{195{,}426{,}000}$ Ton
Corn	$\frac{93{,}347}{1{,}410{,}000}$ Bushel	$\frac{150{,}953}{1{,}410{,}000}$ Bushel
Outputs	$\frac{377}{14{,}100}$ Ton Steel	$\frac{13{,}723}{14{,}100}$ Bushels Corn

Capital per Worker: $\left(\frac{377}{14{,}100}\right.$ Ton, $\left.\frac{2{,}443}{14{,}100}\right.$ Bushel$)$

Net Output Per Worker: $\frac{4}{5}$ Bushel Corn

[34] A common generalization is to consider steady state quantity flows in which the labor force and output grow at a steady state. Then steady state consumption per person-year is a function of the rate of growth, and this function is identical to the frontier relating the wage to the steady state rate of profits.

Table 3-2: Quantity Flows Per Person-Year for the Beta Technique

Inputs	Steel Industry	Corn Industry
Labor	$\frac{573}{4,700}$ Person-Year	$\frac{4,127}{4,700}$ Person-Year
Steel	$\frac{819,221}{65,142,000}$ Ton	$\frac{1,555,879}{65,142,000}$ Ton
Corn	$\frac{703}{470,000}$ Bushel	$\frac{45,397}{470,000}$ Bushel
Outputs	$\frac{377}{10,340}$ Ton Steel	$\frac{4,127}{4,700}$ Bushels Corn

Capital per Worker: $\left(\frac{377}{10,340}\text{ Ton, }\frac{461}{4,700}\text{ Bushel}\right)$

Net Output Per Worker: $\frac{39}{50}$ Bushel Corn

Section 3.1 shows how to determine the cost minimizing technique and prices for a specified rate of profits, while this section specifies stationary-state quantity flows for each technique. These specifications allow one to graph prices versus quantities, normalized in some sense. For example, Figure 3-3 graphs the price of labor, that is, the wage, against the labor-intensity of the cost-minimizing technique at that wage. Labor intensity is measured by the person-years employed per bushel corn produced net. Note that in a comparison of steady states, a higher wage can be associated with a willingness of firms to employ more labor for a given output. This conclusion does not depend on sticky or rigid wages or prices, asymmetrical information, monitoring costs or any other imperfection economists have analyzed in recent decades to explain the failure of capitalist economies to quickly converge to full employment equilibria.

In defining a stationary state, the initial quantities of the capital goods are found by solving the model. These initial quantities are not givens. They are adjusted to the size of the labor force, which is a given parameter of the model. This property of the endogenous determination of the quantities of capital goods contrasts with traditional long-period neoclassical models:

> "Both classical and marginalist economists provided accounts of the long-period (uniform rate of profit) theory of value and distribution, but whereas a classical economist could take the real wage as a datum for the purpose of such analysis (whatever the implicit 'background' theory of wages might be), the marginalist economist had to 'close the system' in some other manner. In effect, since 'resource supplies' were often taken as given, this meant that the 'supply of capital' had to be taken as given, *in one way or another*. Just how the given supply of capital was to be represented was an issue which led to considerable heterogeneity amongst even those marginalist economists who shared the long-period method with the classical economists and with each other… [I]t is now

widely recognized that each version of such traditional long-period marginalist theory of value and distribution encountered insoluble problems." (Steedman 1998)

One division among traditional neoclassical theorists is between those who took capital as a given quantity of value that could vary in form[35] and those who took the quantities of individual capital goods as given in a long period approach[36]. Neoclassical theories constructed on either basis have been shown to be incorrect.[37]

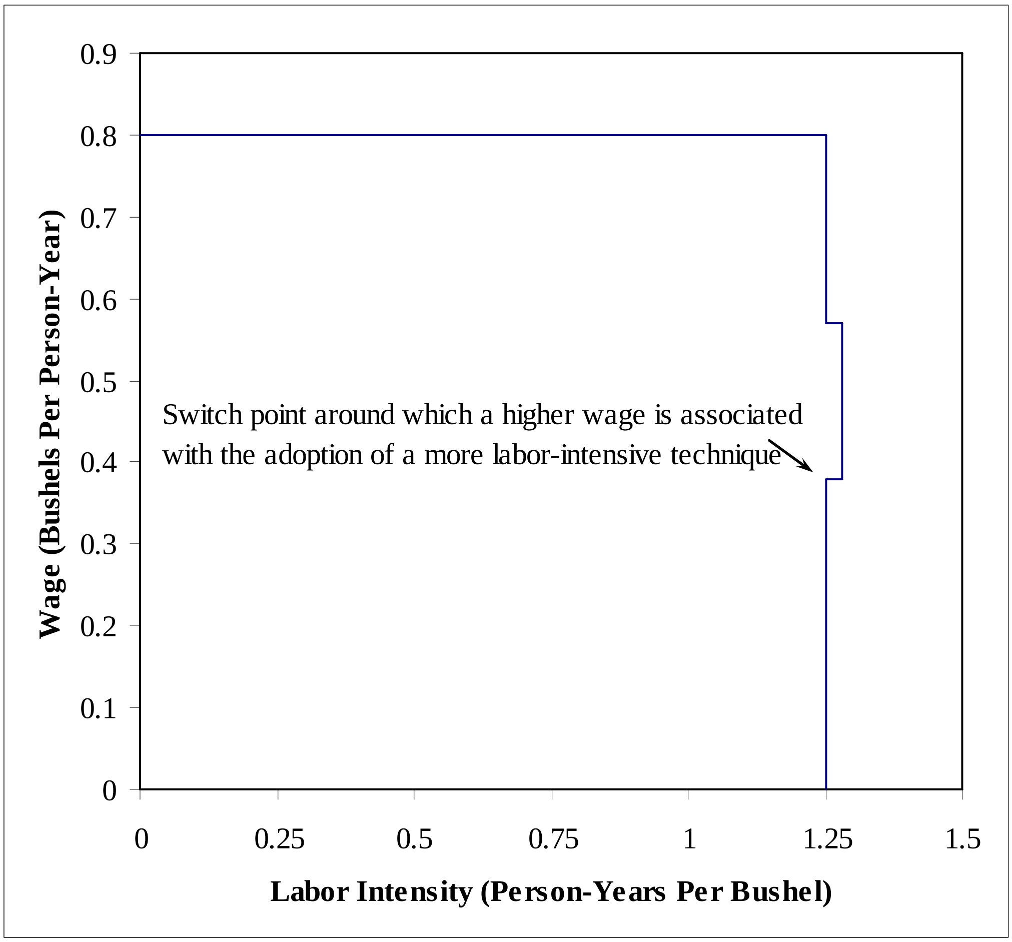

Figure 3-3: Labor Intensity of the Cost-Minimizing Technique

3.3 Stationary State Equilibrium in the Labor and Capital Markets

Imposing equilibrium conditions on the labor and capital markets closes this model of a stationary state. The stationary state employment, which appears in Equations 3-18, 3-19, and 3-20, is equal to the quantity of labor supplied by households in a stationary state. Equation 3-21 specifies the stationary state labor supply:

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[35] For example, Knut Wicksell.
[36] For example, Leon Walras.
[37] See Garegnani (1990b) for an extensive critique.

$$l^*(1+r^*;\gamma,\delta) = 2 - \frac{[1+(1+r^*)][1+\delta+(1+r^*)]\gamma}{(2+\delta)(1+r^*)(1+\gamma)}, \ \eta = \alpha, \beta \quad (3\text{-}21)$$

Gross investment in any year is the value, in bushels corn, of the gross outputs not consumed. Equation 3-22 specifies the stationary state value of gross investment per labor-year:

$$\begin{aligned}\frac{i^*(1+r^*;\gamma,\delta)}{l^*(1+r^*;\gamma,\delta)} &= \mathbf{p}^* \cdot \left[\frac{1}{l^*(1+r^*;\gamma,\delta)}\mathbf{q}^* - \frac{c^*}{l^*(1+r^*;\gamma,\delta)}\mathbf{e}_2\right] \\ &= \frac{a_{01}^{\eta}a_{12}^{\eta} + a_{02}^{\eta}a_{22}^{\eta} - a_{02}^{\eta}(a_{11}^{\eta}a_{22}^{\eta} - a_{12}^{\eta}a_{21}^{\eta})}{\left[a_{12}^{\eta}a_{01}^{\eta} + (1-a_{11}^{\eta})a_{02}^{\eta}\right]\left[(a_{01}^{\eta}a_{12}^{\eta} - a_{02}^{\eta}a_{11}^{\eta})(1+r^*) + a_{02}^{\eta}\right]} \\ &\quad - \frac{(a_{11}^{\eta}a_{22}^{\eta} - a_{12}^{\eta}a_{21}^{\eta})(1+r^*)}{(a_{01}^{\eta}a_{12}^{\eta} - a_{02}^{\eta}a_{11}^{\eta})(1+r^*) + a_{02}^{\eta}}, \ \eta = \alpha, \beta \quad (3\text{-}22)\end{aligned}$$

Like the labor supply, the stationary state savings function is derived from the household's utility-maximization problem. Equation 3-23 specifies stationary state savings:

$$s^*(1+r^*;\gamma,\delta) = \left[1 - \frac{(1+\delta)[1+(1+r^*)]}{(2+\delta)(1+r^*)}\right]w^* \quad (3\text{-}23)$$

Accordingly, Equation 3-24 gives stationary state savings per person-year:

$$\begin{aligned}\frac{s^*(1+r^*;\gamma,\delta)}{l^*(1+r^*;\gamma,\delta)} &= \left\{\frac{(1+\gamma)\left[(1+\delta)-(1+r^*)\right]}{\gamma(1+r^*)^2 - (2+\gamma)(2+\delta)(1+r^*) + (1+\delta)\gamma}\right\}w^* \\ &= \left\{\frac{(1+\gamma)\left[(1+\delta)-(1+r^*)\right]}{\gamma(1+r^*)^2 - (2+\gamma)(2+\delta)(1+r^*) + (1+\delta)\gamma}\right\} \\ &\quad \times \left[\frac{(a_{11}^{\eta}a_{22}^{\eta} - a_{12}^{\eta}a_{21}^{\eta})(1+r^*)^2 - (a_{11}^{\eta} + a_{22}^{\eta})(1+r^*) + 1}{(a_{12}^{\eta}a_{01}^{\eta} - a_{11}^{\eta}a_{02}^{\eta})(1+r^*) + a_{02}^{\eta}}\right], \ \eta = \alpha, \beta \quad (3\text{-}24)\end{aligned}$$

Determining for what rate(s) of profits desired gross savings is equal to gross investment closes this model of stationary states. In other words, one imposes the condition that agents in the model want to hold the capital goods.

$$\frac{s^*(1+r^*;\gamma,\delta)}{l^*(1+r^*;\gamma,\delta)} = \frac{i^*(1+r^*;\gamma,\delta)}{l^*(1+r^*;\gamma,\delta)} \quad (3\text{-}25)$$

The above is a cubic equation in $1+r^*$. Some of the roots of this equation are equilibrium rate of profits, given the values of the parameters, γ and δ, of the utility function. That is, stationary states for different values of the parameters of the utility function can be compared by combining the solution of this cubic equation with the analysis of the choice of technique based on the wage-rate of profits frontier.

Figure 3-4 graphs, for the numerical example and selected values of the utility function parameters, the investment and savings function per person-year. The scaled investment function (Equation 3-22) is independent of the utility function parameters. Two scaled savings functions are shown, corresponding to two sets of utility function parameters.[38] The switch point at $r^* = \frac{5}{4}$ is "perverse", according to traditional neoclassical intuition. Around this switch point, as can be seen in Figure 3-4, a higher interest rate is associated with the adoption of a more capital-intensive technique. This apparent "paradox" is known as *capital reversing*. In the numeric example, the normal switch point at $r^* = \frac{1}{4}$ is associated with a single stationary-state equilibrium. Multiple equilibria arise for utility function parameter values in which one equilibrium is at the "perverse" switch point.

Table 3-3: Some Equilibria for the Example

Utility Function Parameters		**Rate Of Profits**	**Cost-Minimizing Technique**
δ	γ		
$\frac{97}{4852}$	3	75%	Alpha
$\frac{89}{1556}$	3	75%	Beta
$\frac{451}{4292}$	$\frac{2381}{819}$	125%	Beta
$\frac{451}{4292}$	3	125%	Alpha
$\frac{1}{8}$	3	125%	Beta

[38] The rightmost savings function results from utility function parameters that are the mean of the parameter values shown in the first two rows of Table 3-3. The leftmost savings function results from utility function parameters that are the mean of the parameter values shown in the last two rows.

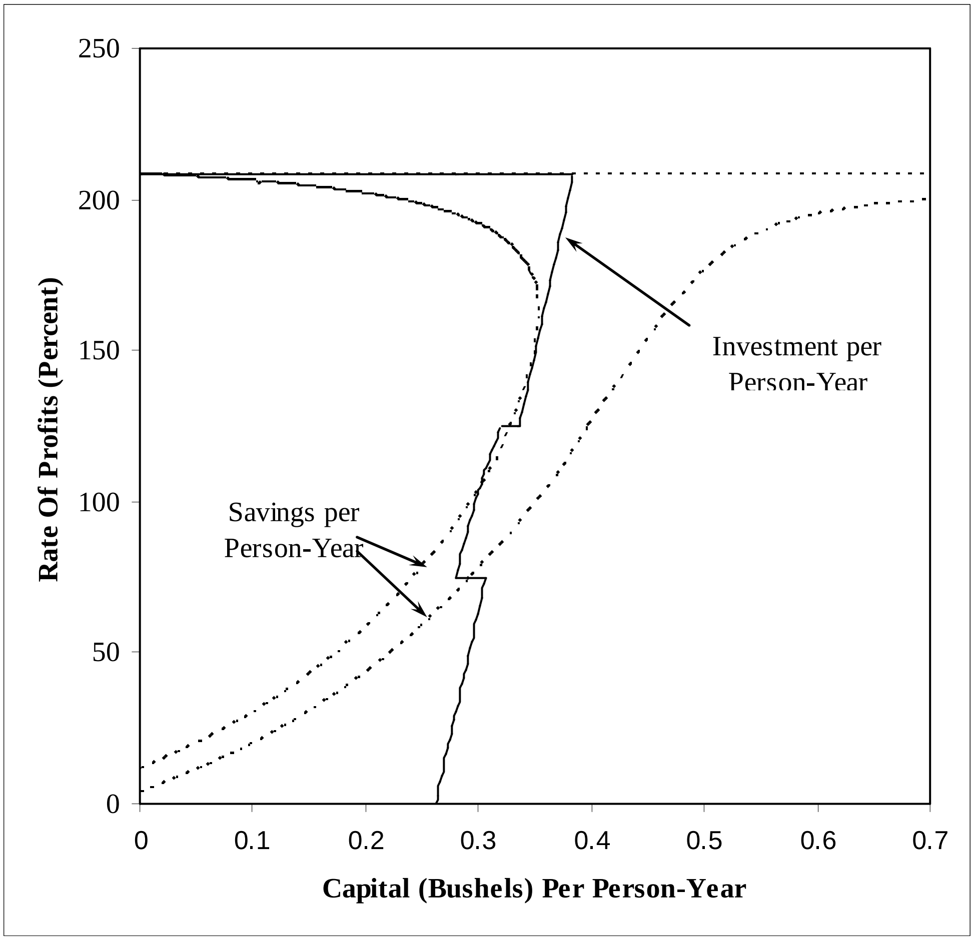

Figure 3-4: Equilibria in the Capital Market

Which switch point is associated with multiple equibria is sensitive to the details of how consumption is modeled. Bloise and Richlin (2005) model consumption in a two-period overlapping generations model with a representative agent. But, unlike the model in this paper, they do not include a choice between labor and consumption in each period. The agent is constrained to supply a person-year of labor in the first period and is retired in the second period. These modeling choices result in multiple equilibria at the normal switch point in the example and a single equilibrium at the "perverse" switch point. Perhaps these implications about multiple equilibrium are tied to the stability of either dynamic equilibrium paths or tatonnement processes.

Various graphs help visualize the stationary state equilibria associated with any exogenously specified parameter values for the utility function. Figure 3-5 shows the equilibrium rates of profits versus the utility function parameter δ; the parameter γ is three for every point on this graph. Recall a lower value of δ indicates a willingness on the part of households to defer consumption, that is, a greater supply of capital, in some sense. Yet a greater supply of capital need not be associated with an equilibrium with a lower equilibrium rate of profits. On the other hand, Figure 3-6 graphs the equilibrium wage versus the utility function parameter γ, where γ controls the household's relative preference for consumption and leisure. A smaller value of γ is associated with a greater willingness to supply labor. Figure 3-6 is drawn for $\delta = \frac{451}{4292}$. Around the switch point with a wage of $\frac{19}{50}$, a greater supply of labor is associated with a higher equilibrium wage, as can be seen in the figure.

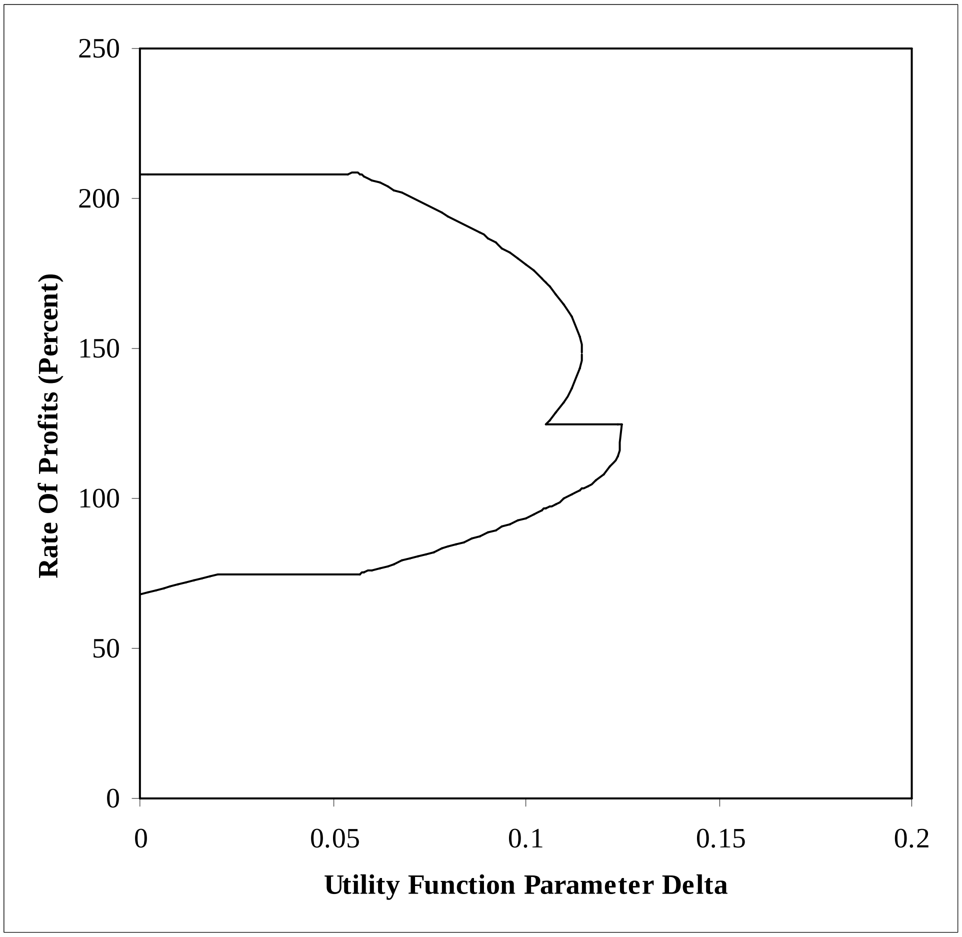

Figure 3-5: Variation in Equilibrium Rate Of Profits with Willingness to Defer Consumption

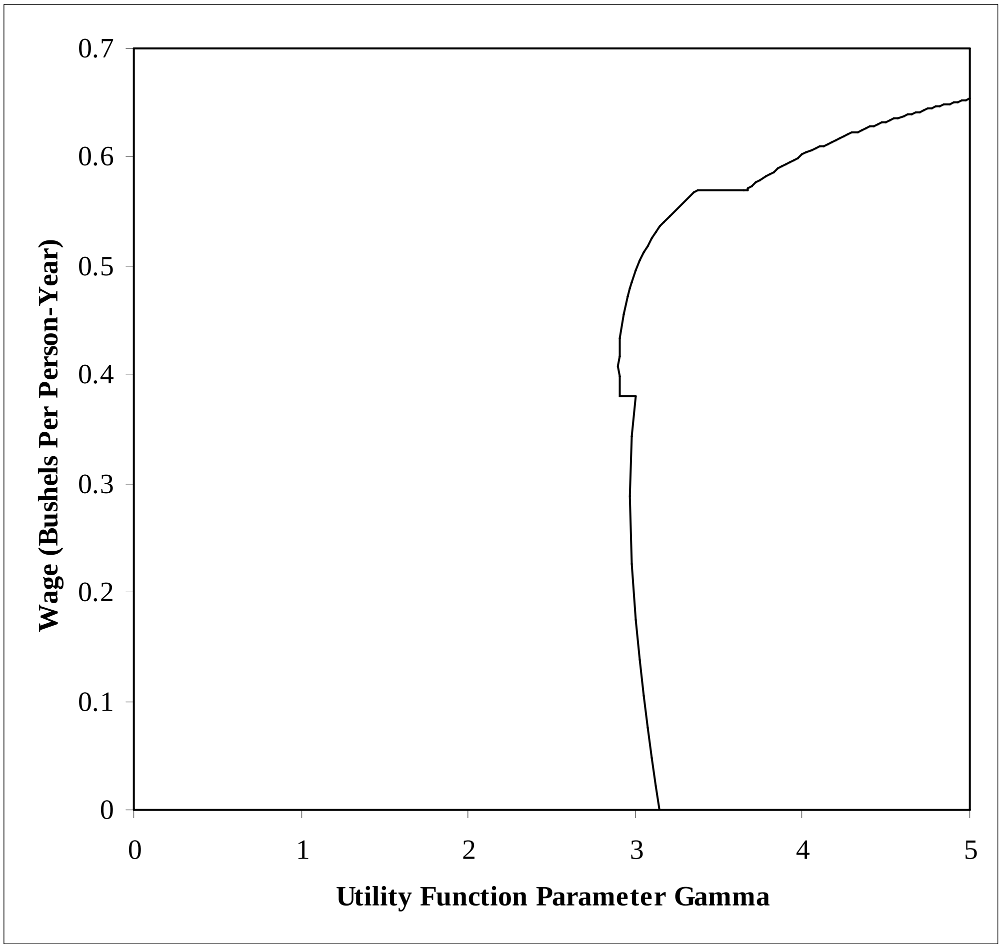

Figure 3-6: Variation in Equilibrium Wage with Preference For Leisure

The example presented in this section does not exhaust the range of stationary state outcomes. Figure 3-7 presents, by graphs analogous to Figures 3-5 and 3-6, two more examples. The example on the left contains a single switch point. This switch point is non "perverse" in that the adoption of a more capital-intensive technique around the switch point is associated with a lower stationary state equilibrium rate of profits. The convexity of the wage-rate of profits curves varies between the techniques. As a consequence, the amount of steel used per bushel corn produced net is not a single-valued function of the price of steel. In a comparison of stationary states, the use of the more steel-intensive technique can be associated with a lower price of steel.

The convexity of the wage-rate of profits curve also varies between two techniques in the reswitching example depicted in Figure 3-7 on the right. This variation in convexity is not possible in two-commodity reswitching examples. The example is a three-commodity model with a structure often used in the theory of capital. As in the example in this section, one switch point is "perverse" in the capital and labor markets in this three commodity example.

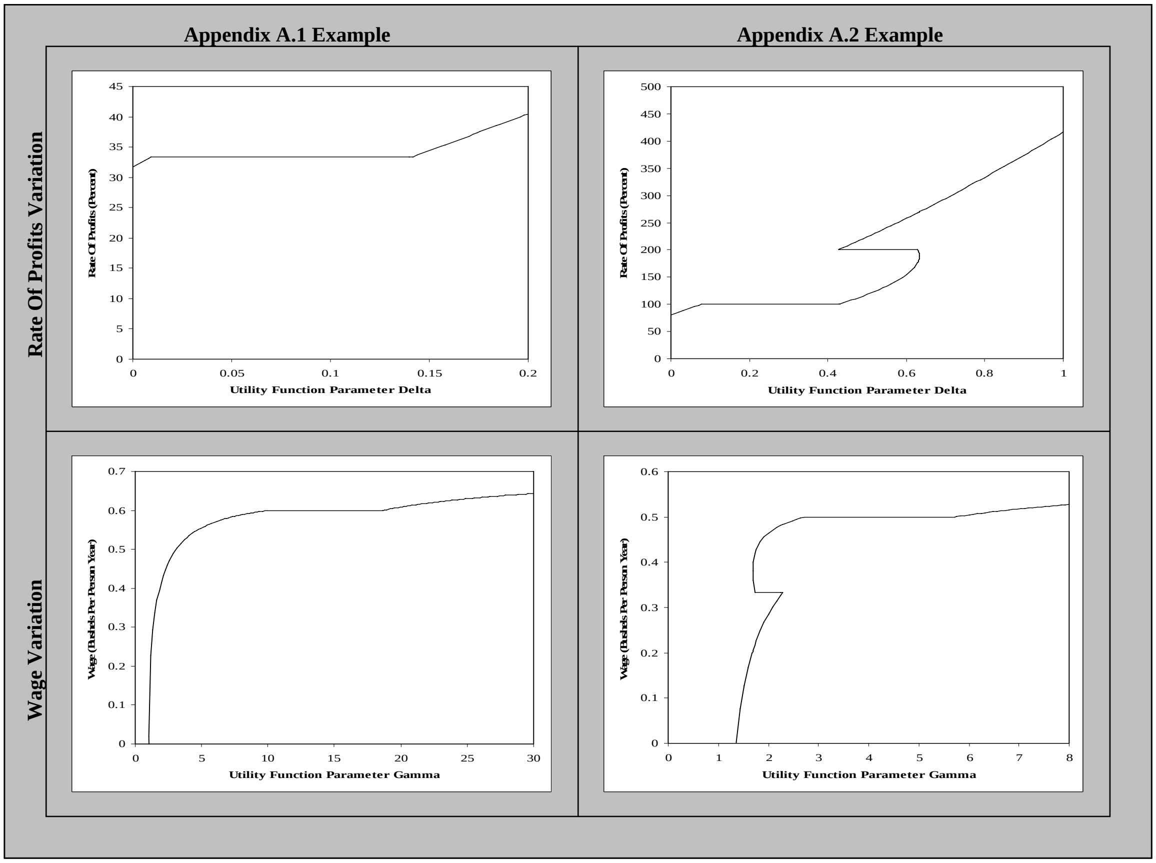

Figure 3-7: Stationary State Examples

3.4 A Program for Future Research

Some interesting dynamic equilibrium paths can be constructed in the example in this section, if stationary states provide guidance to *some* such paths. Parameter values are of interest in which stationary states correspond to the switch point where the rate of profits is 125%. Consider a dynamic equilibrium path for which $\gamma(t) = 3$ for all *t*, $\delta(0) = 1/8 - \varepsilon$, and after some time t_1, $\frac{451}{4292} + \varepsilon = \delta(t_1) = \delta(t_1 + 1) = \ldots$. Presumably, one can find a traverse here in which the economy starts out near a stationary state with the beta technique being cost-minimizing. The path would approach a stationary state with the alpha technique being cost-minimizing. So the result of households supplying more capital would be the adoption of the more capital-intensive technique, as expected. But the rate of profits would *rise* along such a path, as capital becomes less scarce. The construction of such a path would show the neoclassical intuition of the interest rate as a scarcity index for capital is unfounded.

Consider a dynamic equilibrium path for which $\delta(t) = \frac{451}{4292}$ for all *t*, $\gamma(0) = 3 - \varepsilon$, and after some time t_1, $\frac{2381}{819} - \varepsilon = \gamma(t_1) = \gamma(t_1 + 1) = \ldots$. Presumably, one can find a traverse here in which the economy starts out near a stationary state with the alpha technique being cost-minimizing. The path would approach a stationary state with the beta technique being cost-minimizing. So the result of households supplying more labor would be the adoption of the more labor-intensive technique, as expected. But the wage would *rise* along such a path, as labor becomes less scarce. The construction of such a path would show the neoclassical intuition of the wage as a scarcity index for labor is unfounded.

I need to examine some more mathematical economics to complete this critique. One would like to find some reason to believe stationary states have at least saddle point stability. One would like to explicitly construct the dynamic equilibrium paths mentioned above. One might look at additional examples of the choice of technique. And, finally, one would want to know the implications of these results for tatonnement stability, currently a contentious issue in the literature.

Appendix A: Some Stationary State Examples

Figure 3-7 in the main text summarizes stationary states in two examples of the choice of technique. This example provides details for these examples.

A.1 An Example Without Capital Reversing

The left-hand side of Figure 3-7 is derived from an example with a choice of two techniques. One switch point exists in this example, and that switch point does not exhibit capital reversing. The amount of steel used per bushel corn produced net is not a single-valued function of the price of steel in this example.

In analogy with Displays 2-1 and 2-3, Displays A-1 and A-2 specify the alpha technique for this example:

$$\mathbf{a}_0^{\alpha} = \begin{bmatrix} a_{01}^{\alpha} & a_{02}^{\alpha} \end{bmatrix} = \begin{bmatrix} \dfrac{3,220}{3,321} & \dfrac{3,115}{3,321} \end{bmatrix} \tag{A-1}$$

$$\mathbf{A}_{\alpha} = \begin{bmatrix} a_{11}^{\alpha} & a_{12}^{\alpha} \\ a_{21}^{\alpha} & a_{22}^{\alpha} \end{bmatrix} = \begin{bmatrix} 0 & \dfrac{1}{2} \\ \dfrac{1}{18} & 0 \end{bmatrix} \tag{A-2}$$

Displays A-3 and A-4 specify the beta technique:

$$\mathbf{a}_0^{\beta} = \begin{bmatrix} a_{01}^{\beta} & a_{02}^{\beta} \end{bmatrix} = \begin{bmatrix} \dfrac{13,930}{63,099} & \dfrac{3,115}{3,321} \end{bmatrix} \tag{A-3}$$

$$\mathbf{A}_\beta = \begin{bmatrix} a_{11}^\beta & a_{12}^\beta \\ a_{21}^\beta & a_{22}^\beta \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{2,752}{7,011} & 0 \end{bmatrix} \tag{A-4}$$

Note that the corn-producing process is the same in these two techniques. The process for producing the first commodity, steel, differs between the two techniques.

Stationary state quantity flows are easily found for each technique. Table A-1 shows quantity flows per person-year for the alpha technique. Likewise, Table A-2 shows stationary state quantity flows per person-year for the beta technique. Stationary state quantity flows under the beta technique require greater inputs of steel per bushel corn of net output than under the alpha technique.

Price relations can be derived for this example. The beta technique is cost minimizing for rates of profits below $33.\overline{3}$%, while the alpha technique is cost minimizing for rates of profit between $33.\overline{3}$% and 500%. The relations in Displays A-5 and A-6 are found as the solutions to the systems of Equations in Display 3-9. Display A-5 shows the wage-rate of profits frontier:

$$w^* = \begin{cases} \frac{9(5,635 - 2,752 r^* - 1,376 r^{*2})}{35(1,890 + 199 r^*)}, & 0 \leq r^* \leq \frac{1}{3} \\ \frac{369(35 - 2r^* - r^{*2})}{140(135 + 46 r^*)}, & \frac{1}{3} \leq r^* \leq 5 \end{cases} \tag{A-5}$$

The wage-rate of profits frontier is graphed in Figure A-1. Display A-6 shows the price of steel as a function of the rate of profits:

Table A-1: Quantity Flows for the Alpha Technique for Section A.1 Example

Inputs	**Steel Industry**	**Corn Industry**
Labor	$\frac{46}{135}$ Person-Year	$\frac{89}{135}$ Person-Year
Steel	0 Ton	$\frac{123}{350}$ Ton
Corn	$\frac{41}{2,100}$ Bushel	0 Bushel
Outputs	$\frac{123}{350}$ Ton Steel	$\frac{123}{175}$ Bushels Corn

Capital per Worker: $\left(\frac{123}{350}\right.$ Ton, $\frac{41}{2,100}$ Bushel$\left.\right)$

Net Output Per Worker: $\frac{41}{60}$ Bushel Corn

Table A-2: Quantity Flows for the Beta Technique for Section A.1 Example

Inputs	**Steel Industry**	**Corn Industry**
Labor	$\frac{199}{1,890}$ Person-Year	$\frac{1,691}{1,890}$ Person-Year
Steel	0 Ton	$\frac{2,337}{4,900}$ Ton
Corn	$\frac{688}{3,675}$ Bushel	0 Bushel
Outputs	$\frac{2,337}{4,900}$ Ton Steel	$\frac{2,337}{2,450}$ Bushels Corn

Capital per Worker: $\left(\frac{2,337}{4,900}\text{ Ton}, \frac{688}{3,675}\text{ Bushel}\right)$

Net Output Per Worker: $\frac{23}{30}$ Bushel Corn

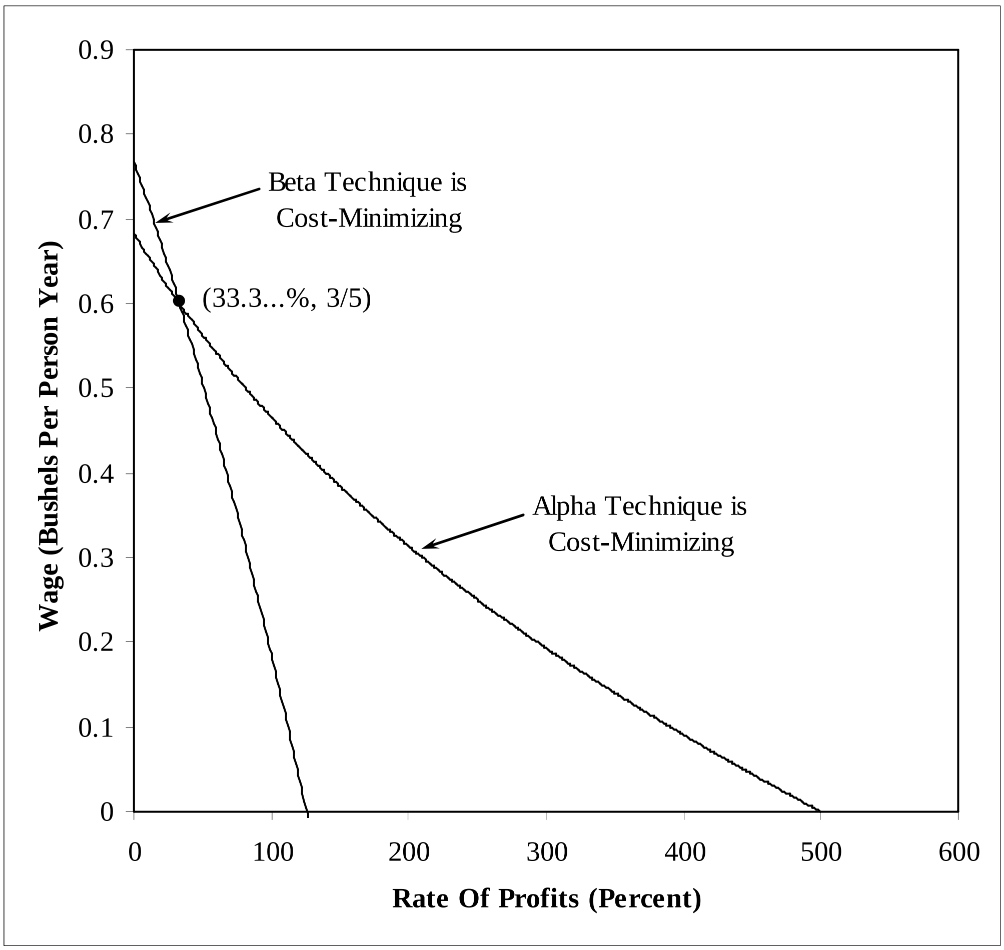

Figure A-1: Wage-Rate of Profits Frontier for Section A.1 Example

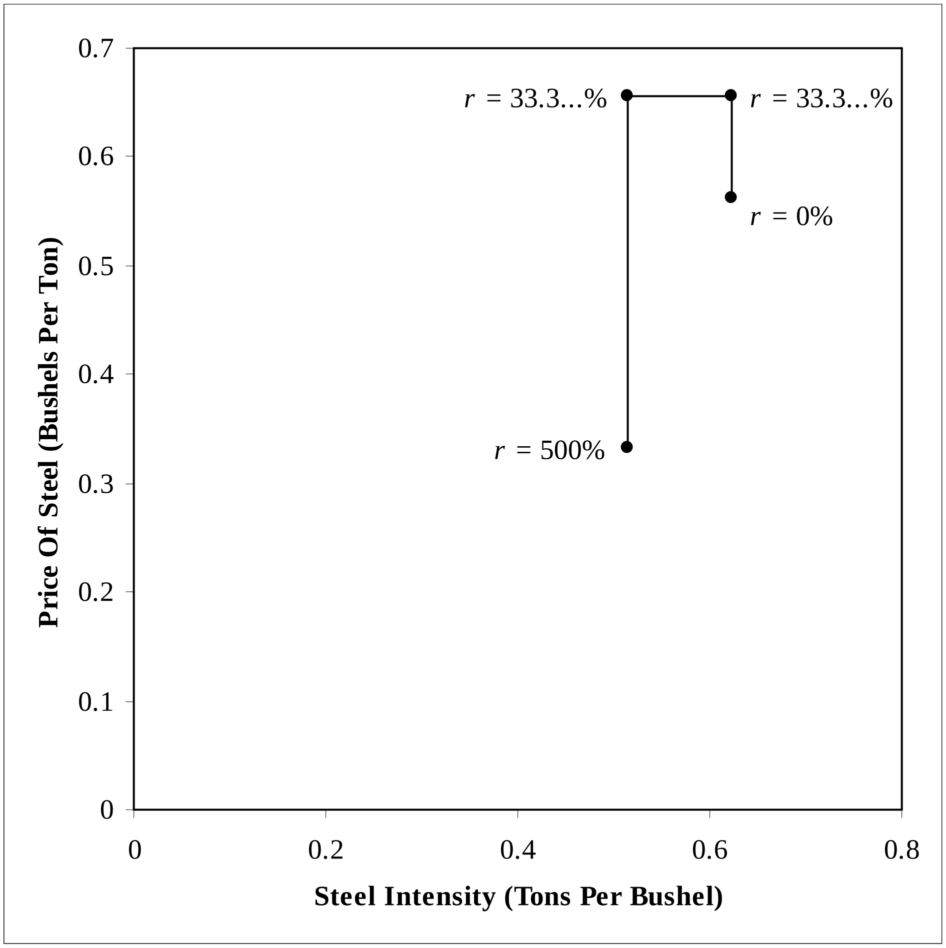

Figure A-2: Price of Steel for Section A.1 Example

$$p_1^* = \begin{cases} \dfrac{2(195{,}895 + 122{,}464r^*)}{369(1{,}890 + 199r^*)}, & 0 \le r^* \le \dfrac{1}{3} \\ \dfrac{1{,}745 + 89r^*}{18(135 + 46r^*)}, & \dfrac{1}{3} \le r^* \le 5 \end{cases} \quad \text{(A-6)}$$

Figure A-2 graphs the price of steel against the steel intensity of the cost minimizing technique.

Equating gross investment and desired savings per person year closes the example. Equation 3-22 specifies gross investment per person year. Equation A-7 shows this specification specialized for this example:

$$\frac{i^*(1+r^*;\gamma,\delta)}{l^*(1+r^*;\gamma,\delta)} = \begin{cases} \dfrac{180{,}647 + 74{,}304r^*}{210(1{,}890 + 199r^*)}, & 0 \le r^* \le \dfrac{1}{3} \\ \dfrac{41(376 + 27r^*)}{420(135 + 46r^*)}, & \dfrac{1}{3} \le r^* \le 5 \end{cases} \quad \text{(A-7)}$$

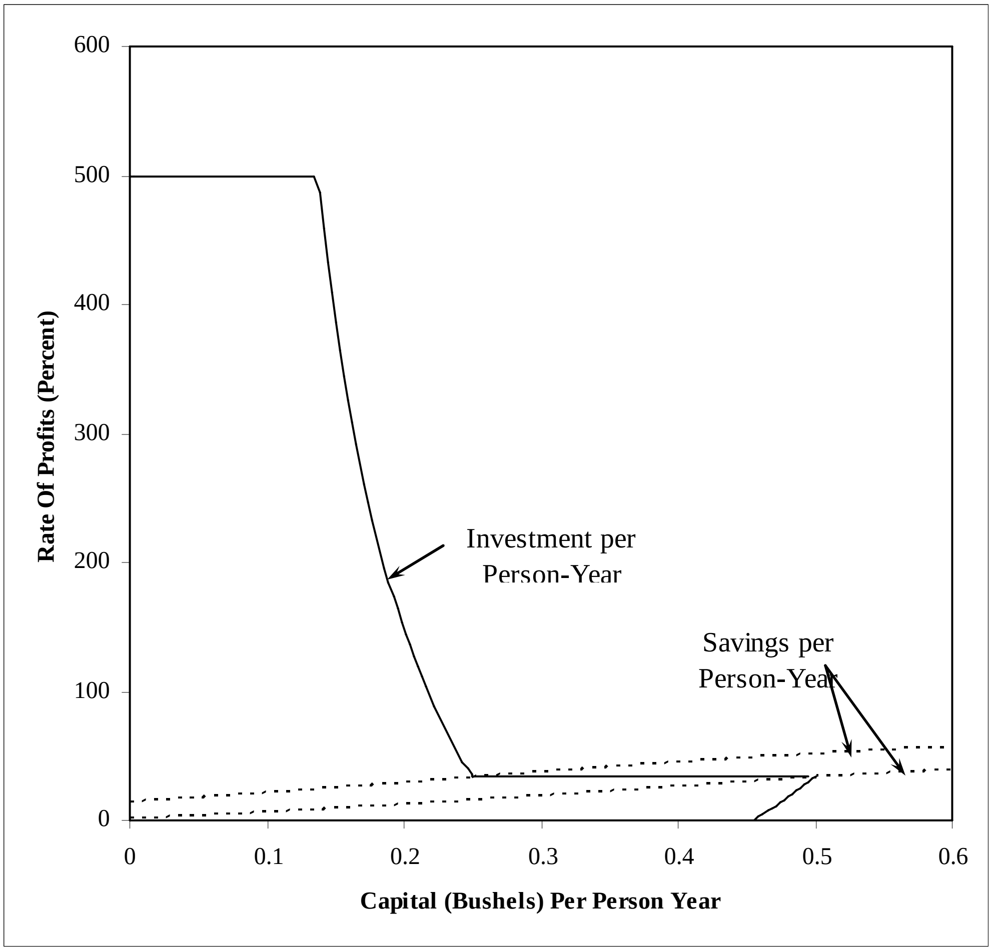

Figure A-3: Equilibria in the Capital Market for Section A.1 Example

Investment and savings functions are graphed in Figure A-3[39].

A.2 A Three-Commodity Reswitching Example

The right-hand side of Figure 3-7 comes from a reswitching example in which firms choose between two techniques. As in the example in Appendix A.1, the convexity of the wage-rate of profits curves varies between the techniques. Such varying convexity in reswitching examples can only be obtained by going beyond two-commodity examples. This is a three-commodity example. Each technique is characterized by the use of a separate capital good, which is used, along with labor, to produce more of that capital good and a net output of the consumption good. Presumably, a traverse between techniques would use an unmodeled backstop technology that allows one capital good to be produced with the aid of the other.

[39] For both savings functions, $\gamma = 10$. The lower savings function corresponds to $\delta = 1/108$. The upper savings function corresponds to $\delta = 103/729$.

The technology is specified by Displays A-8 through A-11. Displays A-8 and A-9 give the coefficients of production for the alpha technique:

$$\mathbf{a}_0^{\alpha} = \begin{bmatrix} a_{01}^{\alpha} & a_{02}^{\alpha} & a_{03}^{\alpha} \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & 0 & 1 \end{bmatrix} \quad \text{(A-8)}$$

$$\mathbf{A}_{\alpha} = \begin{bmatrix} a_{11}^{\alpha} & a_{12}^{\alpha} & a_{13}^{\alpha} \\ a_{21}^{\alpha} & a_{22}^{\alpha} & a_{23}^{\alpha} \\ a_{31}^{\alpha} & a_{32}^{\alpha} & a_{33}^{\alpha} \end{bmatrix} = \begin{bmatrix} \frac{1}{6} & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{(A-9)}$$

Displays A-10 and A-11 give the coefficients of production for the beta technique:

$$\mathbf{a}_0^{\beta} = \begin{bmatrix} a_{01}^{\beta} & a_{02}^{\beta} & a_{03}^{\beta} \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2} & \frac{3}{2} \end{bmatrix} \quad \text{(A-10)}$$

$$\mathbf{A}_{\beta} = \begin{bmatrix} a_{11}^{\beta} & a_{12}^{\beta} & a_{13}^{\beta} \\ a_{21}^{\beta} & a_{22}^{\beta} & a_{23}^{\beta} \\ a_{31}^{\beta} & a_{32}^{\beta} & a_{33}^{\beta} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & 0 \end{bmatrix} \quad \text{(A-11)}$$

Tables A-3 and A-4 specify the quantity flows per person year for the alpha and beta techniques, respectively. Display A-12 specifies the wage-rate of profits frontier for this example:

Table A-3: Quantity Flows for the Alpha Technique for Section A.2 Example

Inputs	**Steel Industry**	**Corn Industry**
Labor	$\frac{2}{7}$ Person-Year	$\frac{5}{7}$ Person-Year
Steel	$\frac{1}{7}$ Ton	$\frac{5}{7}$ Ton
Corn	0 Bushel	0 Bushel
Outputs	$\frac{6}{7}$ Ton Steel	$\frac{5}{7}$ Bushels Corn

Capital per Worker: ($\frac{6}{7}$ Ton Seel, 0 Bushel)

Net Output Per Worker: $\frac{5}{7}$ Bushel Corn

Table A-4: Quantity Flows for the Beta Technique for Section A.2 Example

Inputs	**Tin Industry**	**Corn Industry**
Labor	$\frac{1}{10}$ Person-Year	$\frac{9}{10}$ Person-Year
Tin	$\frac{1}{20}$ Ton	$\frac{3}{20}$ Ton
Corn	0 Bushel	0 Bushel
Outputs	$\frac{1}{5}$ Ton Tin	$\frac{3}{5}$ Bushels Corn

Capital per Worker: ($\frac{1}{5}$ Ton Tin, 0 Bushel)

Net Output Per Worker: $\frac{3}{5}$ Bushel Corn

$$w^* = \begin{cases} \dfrac{5 - r^*}{7 + r^*}, & 0 \le r^* \le 1 \text{ or } 2 \le r^* \le 5 \\ \dfrac{3 - r^*}{5 - r^*}, & 1 \le r^* \le 2 \end{cases} \quad \text{(A-12)}$$

The frontier is graphed in Figure A-4. Equation A-13 gives the stationary state price of steel when the alpha technique is cost-minimizing:

$$p_1^* = \frac{2}{7 + r^*}, \; 0 \le r^* \le 1 \text{ or } 2 \le r^* \le 5 \quad \text{(A-13)}$$

Equation A-14 gives the stationary state price of tin when the beta technique is cost-minimizing:

$$p_2^* = \frac{2}{5 - r^*}, \; 1 \le r^* \le 2 \quad \text{(A-14)}$$

Equation A-15 specifies gross investment per person year:

$$\frac{i^*(1 + r^*; \gamma, \delta)}{l^*(1 + r^*; \gamma, \delta)} = \begin{cases} \dfrac{12}{7(7 + r^*)}, & 0 \le r^* \le 1 \text{ or } 2 \le r^* \le 5 \\ \dfrac{2}{5(5 - r^*)}, & 1 \le r^* \le 2 \end{cases} \quad \text{(A-15)}$$

Investment and savings functions are graphed in Figure A-5[40].

[40] Both savings functions are graphed for $\gamma = 2$. In the upper savings function, $\delta = 1{,}126/2{,}135$. In the lower savings function, $\delta = 23/91$.

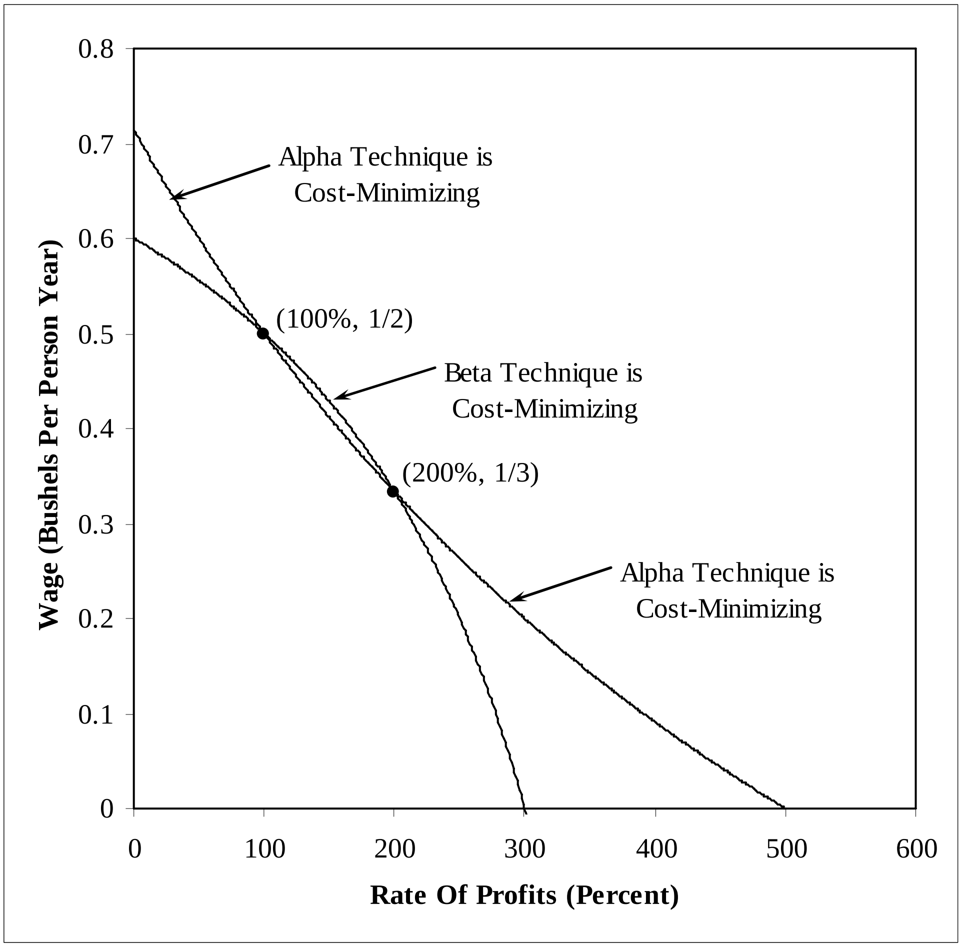

Figure A-4: Wage-Rate of Profits Frontier for Section A.2 Example

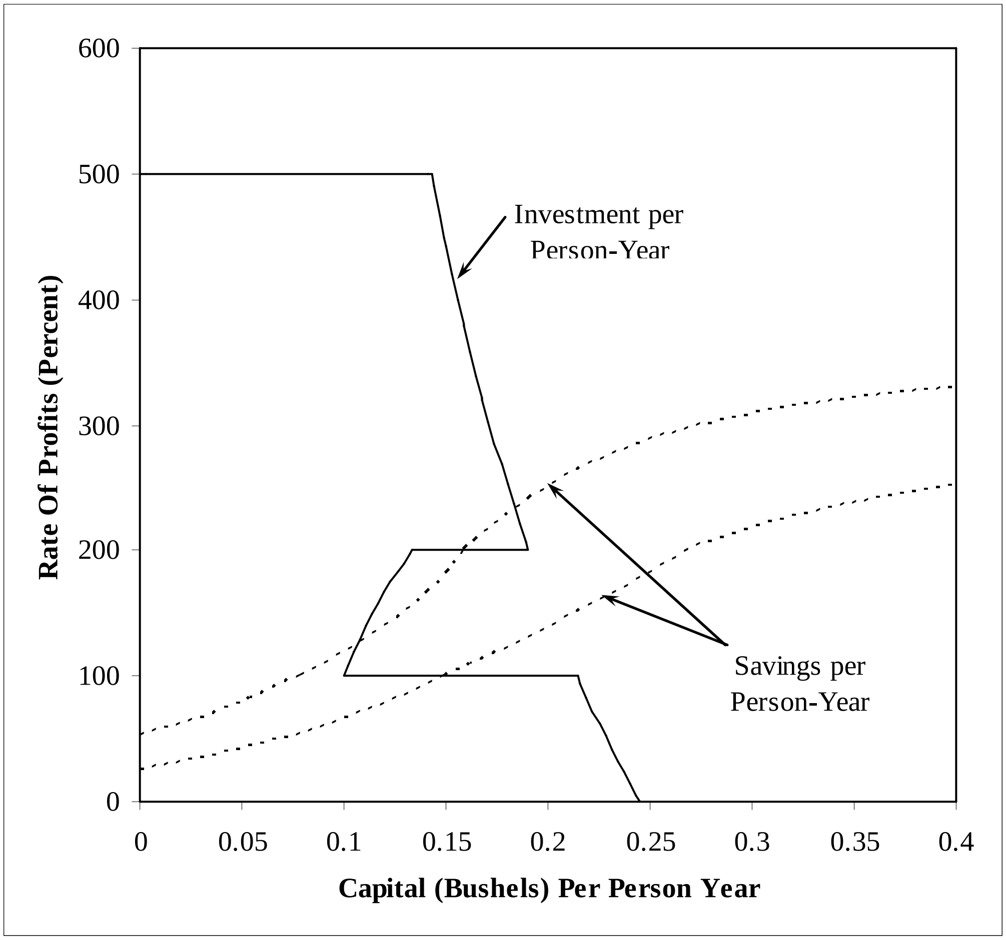

Figure A-5: Equilibria in the Capital Market for Section A.2 Example

Appendix B: Stationary State Savings With Log-Linear Utility Function

Consider an agent, who lives *n* periods, deciding on an intertemporal consumption path. Assume the agent has a log-linear utility function, as in the main text. Given a stationary-state price system, the agent must solve the mathematical programming problem in Display B-1:

$$
\begin{aligned}
&\text{Choose } c_0,\ 1-l_0,\ k_0,\ \ldots,\ c_{n-2},\ 1-l_{n-2},\ k_{n-2},\ c_{n-1},\ 1-l_{n-1} \\
&\text{To maximize } U\left(c_0, 1-l_0, c_1, 1-l_1, \ldots, c_{n-1}, 1-l_{n-1}\right) \\
&\qquad = \sum_{i=0}^{n-1} \frac{1}{(1+\delta)^i}\left[\ln(c_i) + \gamma \ln(1-l_i)\right] \qquad \text{(B-1)} \\
&\text{Such that } c_0 + k_0 = wl_0 \\
&\qquad c_1 + k_1 = wl_1 + (1+r)k_0 \\
&\qquad \vdots \\
&\qquad c_{n-2} + k_{n-2} = wl_{n-2} + (1+r)k_{n-3} \\
&\qquad c_{n-1} = wl_{n-1} + (1+r)k_{n-2} \\
&\qquad c_0 \geq 0,\ l_0 \geq 0,\ k_0 \geq 0,\ c_1 \geq 0,\ l_1 \geq 0,\ k_1 \geq 0,\ \ldots, \\
&\qquad l_{n-2} \geq 0,\ c_{n-2} \geq 0,\ k_{n-2} \geq 0,\ k_{n-2} \geq 0,\ c_{n-1} \geq 0,\ l_{n-1} \geq 0
\end{aligned}
$$

Although l_i is the quantity of labor supplied by the agent in the *i*th year after his birth, the problem is formulated in terms of leisure, $(1-l_i)$. The economy in which this agent is embedded permits savings to be made by purchasing ownership of firms; no borrowing from the future is allowed. Constraining the budget for each year reflects this lack of a market for bonds. As δ becomes smaller, the agent becomes less impatient. As γ becomes smaller, the agent becomes more willing to sacrifice leisure to obtain consumption. That is, a stationary state with a smaller δ or γ characterizes an economy in which households supply a greater quantity of "capital" or labor, respectively.

When the agent is maximizing utility at an interior point solution, the Marginal Rate of Substitution (MRS) between any pair of goods[41] entering the utility function must be equal to the (marginal) rate at which they can be traded off at the prices given in the market, as seen in the budget constraints. Not all of these marginal conditions are independent of one another. I look at the MRS of consumption between successive years:

$$
\frac{\dfrac{\partial U}{\partial c_{i-1}}}{\dfrac{\partial U}{\partial c_i}} = (1+\delta)\frac{c_i}{c_{i-1}} = 1 + r,\ i = 1, 2, \ldots, n-1 \qquad \text{(B-2)}
$$

[41] Goods consist of consumption at the end of each year the agent is alive and leisure during each such year.

and the MRS of leisure and consumption in each given year:

$$\frac{\frac{\partial U}{\partial(1-l_i)}}{\frac{\partial U}{\partial c_i}} = \frac{\gamma c_i}{1-l_i} = w,\ i = 0, 1, \ldots, n-1 \qquad \text{(B-3)}$$

The above derivation yields $(3n-1)$ linear and independent equations (n budget constraints and $(2n-1)$ marginal conditions) in $(3n-1)$ variables (the quantities of corn consumed at the end of each of n years, the leisure time spent not working in each of n years, and the numeraire-value of savings at the end of each of $(n-1)$ years.[42]) The system is easily solved. Multiply the ith constraint by $(1+r)^{n-i}$, thereby obtaining the n equations in Displays B-4, B-5, and B-6:

$$(1+r)^{n-1}c_0 + (1+r)^{n-1}k_0 = w(1+r)^{n-1}l_0 \qquad \text{(B-4)}$$

$$\begin{aligned}(1+r)^{n-i}c_{i-1} &+ (1+r)^{n-i}k_{i-1} \\ &= w(1+r)^{n-i}l_{i-1} + (1+r)^{n-i+1}k_{i-2},\ i = 2, 3, \ldots, n-1\end{aligned} \qquad \text{(B-5)}$$

$$c_{n-1} = wl_{n-1} + (1+r)k_{n-2} \qquad \text{(B-6)}$$

The numeraire-value of savings at the end of each year drops out when these equations are summed:

$$\sum_{i=0}^{n-1}(1+r)^i c_{n-(i+1)} = \sum_{i=0}^{n-1} w(1+r)^i l_{n-(i+1)} \qquad \text{(B-7)}$$

The first set of marginal conditions (Display B-2) allows one to express the agent's consumption at the end of each year in terms of the agent's consumption at the end of the first year:

$$c_i = \frac{(1+r)^i}{(1+\delta)^i}c_0,\ i = 0, 1, \ldots, n-1 \qquad \text{(B-8)}$$

The second set of marginal conditions (Display B-3) allows one to solve for the supply of labor each year in terms of the consumption at the end of that year:

[42] It is assumed that the agent does not save anything at the end of the last year of his life. No bequests are permitted in the model.)

$$l_i = 1 - \frac{\gamma}{w} c_i\,,\ i = 0, 1, \ldots, n-1 \tag{B-9}$$

Substitute into Equation B-9 the expression for consumption given in Equation B-8. One obtains Equation B-10:

$$l_i = 1 - \frac{\gamma}{w} \frac{(1+r)^i}{(1+\delta)^i} c_0\,,\ i = 0, 1, \ldots, n-1 \tag{B-10}$$

Substitute these expressions for yearly consumption (Equation B-8) and labor (Equation B-10) into Equation B-7. Equation B-11 follows:

$$(1+r)^{n-1} c_0 \sum_{i=0}^{n-1} \left(\frac{1}{1+\delta} \right)^i = w \sum_{i=0}^{n-1} (1+r)^i - \gamma (1+r)^{n-1} c_0 \sum_{i=0}^{n-1} \left(\frac{1}{1+\delta} \right)^i \tag{B-11}$$

Recall the sum of a finite geometric series, for example:

$$\sum_{i=0}^{n-1} (1+r)^i = \frac{(1+r)^n - 1}{r} \tag{B-12}$$

Equation B-13 therefore gives each agent’s consumption in a stationary state at the end of each agent’s first year:

$$c_0 = \left[\frac{\delta (1+\delta)^{n-1}}{(1+\delta)^n - 1} \left(\frac{w}{1+\gamma} \right) \frac{(1+r)^n - 1}{r(1+r)^{n-1}} \right] \tag{B-13}$$

The agent’s remaining choice variables are found by substituting B-13 into the relevant equation. Equation B-14 gives the agent’s consumption at the end of each year of his life:

$$c_i = \left[\frac{\delta (1+\delta)^{n-i-1}}{(1+\delta)^n - 1} \left(\frac{w}{1+\gamma} \right) \frac{(1+r)^n - 1}{r(1+r)^{n-i-1}} \right],\ i = 0, 1, \ldots, n-1 \tag{B-14}$$

The agent supplies each year the person-years of labor specified by Equation B-15:

$$l_i = 1 - \left[\frac{\delta (1+\delta)^{n-i-1}}{(1+\delta)^n - 1} \left(\frac{\gamma}{1+\gamma} \right) \frac{(1+r)^n - 1}{r(1+r)^{n-i-1}} \right],\ i = 0, 1, \ldots, n-1 \tag{B-15}$$

Tables B-1 through B-4 summarize the solution to this intertemporal utility-maximization problem for models in which agents live one, two, and three years, respectively.

Table B-1: Intertemporal Stationary-State Labor Supply

n	l_0	l_1	l_2
1	$\frac{1}{1+\gamma}$		
2	$1-\frac{(1+\delta)(2+r)\gamma}{(2+\delta)(1+r)(1+\gamma)}$	$1-\frac{(2+r)\gamma}{(2+\delta)(1+\gamma)}$	
3	$1-\frac{(1+2\delta+\delta^2)(3+3r+r^2)\gamma}{(3+3\delta+\delta^2)(1+2r+r^2)(1+\gamma)}$	$1-\frac{(1+\delta)(3+3r+r^2)\gamma}{(3+3\delta+\delta^2)(1+r)(1+\gamma)}$	$1-\frac{(3+3r+r^2)\gamma}{(3+3\delta+\delta^2)(1+\gamma)}$

Table B-2: Intertemporal Stationary-State Consumption

n	c_0	c_1	c_2
1	$\frac{w}{1+\gamma}$		
2	$\frac{(1+\delta)(2+r)w}{(2+\delta)(1+r)(1+\gamma)}$	$\frac{(2+r)w}{(2+\delta)(1+\gamma)}$	
3	$\frac{(1+2\delta+\delta^2)(3+3r+r^2)w}{(3+3\delta+\delta^2)(1+2r+r^2)(1+\gamma)}$	$\frac{(1+\delta)(3+3r+r^2)w}{(3+3\delta+\delta^2)(1+r)(1+\gamma)}$	$\frac{(3+3r+r^2)w}{(3+3\delta+\delta^2)(1+\gamma)}$

Table B-3: Stationary-State Labor Supply Functions

n	**Labor Supply Function**
1	$L_s=\frac{1}{1+\gamma}$
2	$L_s=2-\frac{(2+r)(2+\delta+r)\gamma}{(2+\delta)(1+r)(1+\gamma)}$
3	$L_s=3-\frac{(3+3r+r^2)(3+3\delta+\delta^2+\delta r+3r+r^2)\gamma}{(3+3\delta+\delta^2)(1+r)^2(1+\gamma)}$

Table B-4: Stationary-State Savings Functions

n	**Savings Function**
1	$S=0$
2	$S=\left[1-\frac{(1+\delta)(2+r)}{(2+\delta)(1+r)}\right]w$
3	$S=\left[3+r-\frac{(3+3r+r^2)(3+5\delta+2\delta^2+2r+3\delta r+\delta^2 r)}{(3+3\delta+\delta^2)(1+r)}\right]w$

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